Covariant amplitudes for mesons

Jie-Jie Zhu*, Mu-Lin Yan
CCAST(World Lab), P. O. Box 8730, Beijing, 100080, P. R. China
and Center for Fundamental Physics, USTC, Hefei, Anhui, 230026, P. R. China†
(February 1, 2008)

We show how to construct covariant amplitudes for processes involving higher spins in this paper. First we give the explicit expressions of Rarita-Schwinger wave functions and propagators for bosons with spins, then kinematic singularity free 3-leg effective vertexes are derived and given in a list. Equivalence relations are worked out to get these independent vertexes. Constraints of space reflection symmetry and boson symmetry are considered and shown in a explicit way. Some helicity amplitudes for two-body decays in center of frame are calculated. Finally the covariant helicity amplitudes for the process $a_1 \to \pi^+\pi^+\pi^-$ are constructed to illustrate how to include background (1PI) amplitudes. Both background amplitudes and resonance amplitudes are needed to give reliable descriptions to high energy reactions.

PACS number(s): 11.80.Cr, 11.55Bq, 13.25-k

I. INTRODUCTION

Model independent amplitudes are needed to analyze high energy experimental data. Such amplitudes are usually written in terms of helicity formalism first proposed by Jacob and Wick [1–4], or tensor formalism by Zemach [5,6], in a non-covariant form. Recently Chung [7] and Filippini *et al.* [8] emphasized on the importance of covariance to get reliable results. Chung gave some examples of two-body decays with their amplitudes calculated in center of mass frame in his works [7,9] on covariant tensor formalism. The case of spin-J $\to$ spin-j + spin-0 with $J, j \le 2$ has been discussed in detail in Ref. [8].

We construct covariant amplitudes in a view of S-matrix approach. We call a S-matrix (maybe off shell) after stripping off external lines (wave functions) effective vertexes. Effective vertexes are related to Green functions by LSZ reduction formulations [10,11], which can be divide into one-particle irreducible (1PI) parts and one-particle reducible parts. The former ones are often called backgrounds and the later resonances. One should repeat the process to divide those sub-vertexes connected by (full) propagators in resonance parts, until arrives at 3-leg vertexes that can not be separated.

These effective vertexes (with three or more legs) should be constructed from the four-momenta of outer legs and isotropic tensors of Lorentz group. We use wave functions satisfying Rarita-Schwinger conditions [12] in this paper. This specific choice of wave functions will not introduce any model dependence, since we can change our results into any other representations via basis transformations. Information on the structure of particles is contained in effective vertexes. The general form of an effective vertex for bosons are tensors to be constructed from p_i^μ $(i = 1, 2, \cdots)$ and $g^{\mu\nu}, \varepsilon^{\alpha\beta\gamma\delta}$. Following the assumption of maximal analyticity [13–15], such vertex functions are free of kinematic singularities (K. S.) [14,16–19]. However, one should be careful when writing down independent effective vertexes. Since there are redundant components to be removed by Rarita-Schwinger conditions, some seemingly independent terms are related by equivalence relations [20]. We work out equivalence relations and give a general list of 3-leg effective vertexes. Some special cases of them have already been given by Scadron [20].

Additional symmetries give constraints on the form factors in effective vertexes. Space reflection symmetry demands effective vertexes being tensors or pseudo-tensors, depending on spin-parities of their outer legs. The ratio of form factors in tensor parts and those in pseudo-tensor parts can be taken as a parameter of parity violation. For 3-leg effective vertexes involving two spin-0 particles, some processes are forbidden.

If an effective vertex is connected to two identical bosons, it must satisfy boson symmetry. There are kinematic zeros in form factors that should be factored out. When both of the two bosons are on shell, the "anti-symmetric" parts of the 3-leg effective vertex vanish. For the case of one particle is on shell while another is off shell, the contribution from "anti-symmetric" parts is not independent from background terms, since the form factors in these parts contribute a

*Electronic address: jjzhu@ustc.edu.cn
†Permanent address.

factor which eliminates the denominator of the propagator for the off shell particle. We give a list of 3-leg effective vertexes, with all these cases considered.

Covariant amplitudes in different reference frame are related by Wigner rotations [21]. We calculate the amplitudes for some two-body decays in center of mass (CM) frame. The relation between amplitudes in laboratory frame and those in CM frame is derived.

Background amplitudes should not be neglected in order to give a full description to a reaction, and to get reliable information from data [22]. 1PI amplitudes will not give flat distributions if there are particles with non-zero spins. The covariant helicity amplitudes for $a_1 \to \pi^+\pi^+\pi^-$ are constructed as a demonstration.

Our present work is partly based on Ref. [23]. In Sec. II we give a brief introduction to wave functions and propagators. 3-leg effective vertexes are listed in Sec. III. Sec. IV and Sec. V are devoted to the constraint of space reflection symmetry and boson symmetry. Two-body decays are considered in Sec. VI. In Sec. VII we discuss the process $a_1 \to \pi^+\pi^+\pi^-$.

II. WAVE FUNCTIONS

Let $L(p)$ be a Lorentz transformation,

$$p^\mu = L^\mu{}_\nu(p)k^\nu. \tag{1}$$

The standard momentum for a mass-W particles is $(k^\mu) = (W;\ \vec{0})$. The space-time metric we use is $(g^{\mu\nu}) = \mathrm{diag}\{1,-1,-1,-1\}$. Now define one-particle states as [24]

$$|p,\sigma\rangle = U(L(p))|k,\sigma\rangle \tag{2}$$

with $U(L(p))$ the unitary representation of $L(p)$ in Hilbert space. It satisfies

$$\hat{p}^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle. \tag{3}$$

We can choose the orthonormal condition to be

$$\langle p^{'},\sigma^{'}|p,\sigma\rangle = (2\pi)^3(2p^0)\delta(\vec{p}^{'} - \vec{p})\delta_{\sigma^{'}\sigma}. \tag{4}$$

Under a Lorentz transformation Λ,

$$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma^{'}} D_{\sigma^{'}\sigma}(W(\Lambda,p))|\Lambda p,\sigma^{'}\rangle, \tag{5}$$

where

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p) \tag{6}$$

is the Wigner rotation [21] and $\{D_{\sigma^{'}\sigma}\}$ furnishes a representation of $SO(3)$.

If we define the Lorentz transformation in Eq. (2) to be a pure Lorentz boost

$$\begin{aligned} L(p) &= L(\vec{p}) \\ &\equiv R(\varphi,\theta,0)L_z(|\vec{p}|)R^{-1}(\varphi,\theta,0), \end{aligned} \tag{7}$$

we obtain canonical states. Here $R(\varphi,\theta,0)$ is the rotation that takes z-axis to the direction of $\vec{p}$, and the boost $L_z(|\vec{p}|)$ takes the four-momentum $(k^\mu) = (W;\ \vec{0})$ to $\left(\sqrt{W^2+\vec{p}^2};\ 0,\ 0,\ |\vec{p}|\right)$. For a particle of spin-j, $\sigma \sim (j,m)$. For canonical states, Eq. (5) becomes

$$U(\Lambda)|\vec{p},j,m\rangle = \sum_{m^{'}} D^j_{m^{'}m}(L^{-1}(\vec{\Lambda p})\Lambda L(\vec{p}))|\vec{\Lambda p},j,m^{'}\rangle. \tag{8}$$

$D^j_{m^{'}m}$ is the ordinary D-function. Especially, under a rotation R,

$$U(R)|\vec{p},j,m\rangle = \sum_{m^{'}} D^j_{m^{'}m}(R)|\vec{Rp},j,m^{'}\rangle. \tag{9}$$

Defining the Lorentz transformation in another way leads to helicity states [1]:

$$\begin{aligned} L(p) &= L(\overrightarrow{p})R^{-1}(\varphi,\theta,0) \\ &\equiv R(\varphi,\theta,0)L_z(|\overrightarrow{p}|). \end{aligned} \tag{10}$$

We have

$$U(\Lambda)|\overrightarrow{p},j,\lambda\rangle = \sum_{\lambda'} D^j_{\lambda'\lambda}(L^{-1}(\overrightarrow{\Lambda}p)\Lambda L(\overrightarrow{p}))|\overrightarrow{\Lambda}p,j,\lambda'\rangle \tag{11}$$

and

$$U(R)|\overrightarrow{p},j,\lambda\rangle = |\overrightarrow{R}p,j,\lambda\rangle. \tag{12}$$

The two kinds of definitions are related to each other,

$$|\overrightarrow{p},j,\lambda\rangle_{helicity} = \sum_m D^j_{m\lambda}(\varphi,\theta,0)|\overrightarrow{p},j,m\rangle_{canonical}. \tag{13}$$

Quantum states in terms of creation and annihilation operators read:

$$|\overrightarrow{p},j,\sigma\rangle = \sqrt{(2\pi)^3 2p^0}\,a^\dagger(\overrightarrow{p},j,\sigma)|0\rangle, \tag{14}$$

where $|0\rangle$ is the vacuum state. We use "σ" to mean that the relation holds for both helicity states and canonical states. Quantum fields for spin-j bosons are constructed as [24]

$$\phi_{\mu_1\mu_2\ldots\mu_j} = \int \frac{d^3p}{\sqrt{(2\pi)^3\,2p^0}}\{\sum_\sigma e_{\mu_1\mu_2\ldots\mu_j}(\overrightarrow{p},j,\sigma)a(\overrightarrow{p},j,\sigma)e^{-ip\cdot x} + \sum_\sigma e^*_{\mu_1\mu_2\ldots\mu_j}(\overrightarrow{p},j,\sigma)a^{c\dagger}(\overrightarrow{p},j,\sigma)e^{ip\cdot x}\}, \tag{15}$$

with $a^{c\dagger}$ the annihilation operator of the antiparticle, and

$$U(\Lambda,a)\phi_{\mu_1\mu_2\ldots\mu_j}U^{-1}(\Lambda,a) = \Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}\ldots\Lambda_{\mu_j}{}^{\nu_j}\phi_{\nu_1\nu_2\ldots\nu_j}(\Lambda x + a). \tag{16}$$

$e_{\mu_1\mu_2\ldots\mu_j}$ is the wave function in momentum space satisfying [24]

$$\Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}\ldots\Lambda_{\mu_j}{}^{\nu_j}e_{\nu_1\nu_2\ldots\nu_j}(\overrightarrow{p},j,\sigma) = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))e_{\mu_1\mu_2\ldots\mu_j}(\overrightarrow{p},j,\sigma'), \tag{17}$$

so its definition is

$$e_{\mu_1\mu_2\ldots\mu_j}(\overrightarrow{p},j,\sigma) = \Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}\ldots\Lambda_{\mu_j}{}^{\nu_j}e_{\mu_1\mu_2\ldots\mu_j}(\overrightarrow{0},j,\sigma). \tag{18}$$

From the following infinitesimal generators of the Lorentz group

$$\begin{aligned} &(J_1{}^\mu{}_\nu) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0 \end{pmatrix},\ (J_2{}^\mu{}_\nu) = \begin{pmatrix} 0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0 \end{pmatrix},\ (J_3{}^\mu{}_\nu) = \begin{pmatrix} 0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0 \end{pmatrix}, \\ &(K_1{}^\mu{}_\nu) = \begin{pmatrix} 0&i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\quad (K_2{}^\mu{}_\nu) = \begin{pmatrix} 0&0&i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0 \end{pmatrix},\quad (K_3{}^\mu{}_\nu) = \begin{pmatrix} 0&0&0&i\\0&0&0&0\\0&0&0&0\\i&0&0&0 \end{pmatrix}, \end{aligned} \tag{19}$$

one can obtain

$$\left(L_z\left(|\overrightarrow{p}|\right)^\mu{}_\nu\right) \equiv e^{-i\alpha K_3} = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0&1&0&0\\0&0&1&0\\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} = \begin{pmatrix} E/W & 0 & 0 & |\overrightarrow{p}|/W \\ 0&1&0&0\\0&0&1&0\\ |\overrightarrow{p}|/W & 0 & 0 & E/W \end{pmatrix}, \tag{20}$$

$$\left(R\left(\varphi,\theta,0\right)^{\mu}{}_{\nu}\right)\equiv\left(\left(e^{-i\varphi J_3}e^{-i\theta J_2}\right)^{\mu}{}_{\nu}\right)=\begin{pmatrix}1&0&0&0\\0&\cos\theta\cos\varphi&-\sin\varphi&\sin\theta\cos\varphi\\0&\cos\theta\sin\varphi&\cos\varphi&\sin\theta\sin\varphi\\0&-\sin\theta&0&\cos\theta\end{pmatrix}. \tag{21}$$

Choose wave functions at rest frame to be

$$\left(e^{\mu}(\overrightarrow{0},0)\right)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{22}$$

$$\left(e^{\mu}(\overrightarrow{0},\pm1)\right)=\mp\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix}, \tag{23}$$

we get the familiar explicit expressions of spin-1 canonical wave functions(E is the energy of the particle)

$$\begin{aligned}(e_c^{\mu}(\overrightarrow{p},0))&=\begin{pmatrix}\frac{|\overrightarrow{p}|}{W}\cos\theta\\\frac{1}{2}\left(\frac{E}{W}-1\right)\sin2\theta\cos\varphi\\\frac{1}{2}\left(\frac{E}{W}-1\right)\sin2\theta\sin\varphi\\\frac{1}{2}\left(\frac{E}{W}-1\right)(1+\cos2\theta)+1\end{pmatrix},\\(e_c^{\mu}(\overrightarrow{p},\pm1))&=\mp\tfrac{1}{\sqrt{2}}\begin{pmatrix}\frac{|\overrightarrow{p}|}{W}\sin\theta e^{\pm i\varphi}\\\left(\frac{E}{W}-1\right)\sin^2\theta\cos\varphi e^{\pm i\varphi}+1\\\left(\frac{E}{W}-1\right)\sin^2\theta\sin\varphi e^{\pm i\varphi}\pm1\\\left(\frac{E}{W}-1\right)\cos\theta\sin\theta e^{\pm i\varphi}\end{pmatrix}\end{aligned} \tag{24}$$

and spin-1 helicity wave functions

$$\begin{aligned}(e_h^{\mu}(\overrightarrow{p},0))&=\begin{pmatrix}\frac{|\overrightarrow{p}|}{W}\\\frac{E}{W}\sin\theta\cos\varphi\\\frac{E}{W}\sin\theta\sin\varphi\\\frac{E}{W}\cos\theta\end{pmatrix},\\(e_h^{\mu}(\overrightarrow{p},\pm1))&=\tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\\mp\cos\theta\cos\varphi+i\sin\varphi\\\mp\cos\theta\sin\varphi-i\cos\varphi\\\pm\sin\theta\end{pmatrix}.\end{aligned} \tag{25}$$

Wave functions for higher integral spin particles can be defined recurrently by

$$e_{\mu_1\mu_2\cdots\mu_j}(\overrightarrow{p},j,\sigma)=\sum_{\sigma'_{j-1},\sigma_j}(j-1,\sigma'_{j-1};1,\sigma_j|j,\sigma)e_{\mu_1\mu_2\cdots\mu_{j-1}}(\overrightarrow{p},j-1,\sigma'_{j-1})e_{\mu_j}(\overrightarrow{p},\sigma_j). \tag{26}$$

Using the following C-G coefficient relation

$$\begin{aligned}&\sum_{\sigma'_3,\sigma'_4,\cdots,\sigma'_n}(j_1,\sigma_1;j_2,\sigma_2|j_1+j_2,\sigma'_3)(j_1+j_2,\sigma'_3;k_3,\sigma_3|j_1+j_2+j_3,\sigma'_4)\cdots\\&\qquad\times(j_1+j_2+\cdots+j_{n-1},\sigma'_n;j_n,\sigma_n|j_1+j_2+\cdots+j_n,\sigma'_n+\sigma_n)\\&=\left[\prod_{i=1}^{n}\tfrac{(2j_i)!}{(j_i+\sigma_i)!(j_i-\sigma_i)!}\right]^{\frac{1}{2}}\left\{\frac{\left[\sum_{i=1}^{n}(j_i+\sigma_i)\right]!\left[\sum_{i=1}^{n}(j_i-\sigma_i)\right]!}{\left(2\sum_{i=1}^{n}j_i\right)!}\right\}^{\frac{1}{2}},\end{aligned} \tag{27}$$

we find

$$\begin{aligned}&e_{\mu_1\mu_2\cdots\mu_j}(\overrightarrow{p},j,\sigma)\\&=\sum_{\sigma_1,\sigma_2,\cdots,\sigma_j}\left\{\frac{2^j(j+\sigma)!(j-\sigma)!}{(2j)!\prod_{i=1}^{j}[(1+\sigma_i)!(1-\sigma_i)!]}\right\}^{\frac{1}{2}}\delta_{\sigma_1+\sigma_2+\cdots+\sigma_j,\sigma}e_{\mu_1}(\overrightarrow{p},\sigma_1)e_{\mu_2}(\overrightarrow{p},\sigma_2)\cdots e_{\mu_j}(\overrightarrow{p},\sigma_j).\end{aligned} \tag{28}$$

The above expression is equivalent to that given by Scadron [20] and Chung [25], since they come from the same definition of Eq. (26).

$e_{\mu_1\mu_2\cdots\mu_j}(\vec{p},j,\sigma)$ satisfies Rarita-Schwinger conditions: space-like, symmetric and traceless:

$$p^{\mu_1}e_{\mu_1\mu_2...\mu_j}(\vec{p},j,\sigma)=0, \tag{29}$$

$$e_{\mu_1...\mu_k...\mu_l...\mu_j}(\vec{p},j,\sigma)=e_{\mu_1...\mu_l...\mu_k...\mu_j}(\vec{p},j,\sigma), \tag{30}$$

$$g^{\mu_1\mu_2}e_{\mu_1\mu_2...\mu_j}(\vec{p},j,\sigma)=0. \tag{31}$$

Spin projection operator is [26]

$$\begin{aligned}
&\mathcal{P}^{(j)}_{\mu_1\mu_2...\mu_j;\nu_1\nu_2...\nu_j}\\
\equiv\ & \sum_\sigma e_{\mu_1\mu_2...\mu_j}(\vec{p},j,\sigma)e^*_{\nu_1\nu_2...\nu_j}(\vec{p},j,\sigma)\\
=\ & \tfrac{1}{(j!)^2}\sum_{\substack{P\{\mu_1\mu_2...\mu_j\}\\ P\{\nu_1\nu_2...\nu_j\}}}\left(\prod_{i=1}^{j}\tilde{g}_{\mu_i\nu_i}+a_1^{(j)}\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\nu_1\nu_2}\prod_{i=3}^{j}\tilde{g}_{\mu_i\nu_i}+...\right.\\
&+\begin{cases} a^{(j)}_{j/2}\prod_{i=1}^{j/2}\left(\tilde{g}_{\mu_{2i-1}\mu_{2i}}\tilde{g}_{\nu_{2i-1}\nu_{2i}}\right)\Bigg), & \text{for even } j,\\ a^{(j)}_{(j-1)/2}\tilde{g}_{\mu_j\nu_j}\prod_{i=1}^{(j-1)/2}\left(\tilde{g}_{\mu_{2i-1}\mu_{2i}}\tilde{g}_{\nu_{2i-1}\nu_{2i}}\right)\Bigg), & \text{for odd } j;\end{cases}
\end{aligned} \tag{32}$$

where

$$a_K^{(j)}=\frac{(-1)^K j!}{2^K K!(j-2K)!}\frac{1}{(2j-1)(2j-3)...(2j-2K+1)}, \tag{33}$$

and

$$\tilde{g}_{\mu\nu}=-g_{\mu\nu}+\frac{p_\mu p_\nu}{W^2} \tag{34}$$

is the spin-1 projection operator. Spin projection operators are useful when we sum over final (or initial) particles' spins. They also serve as numerators of propagators [24]. From Eq. (32), the first five projection operators read

$$\mathcal{P}^{(1)}_{\mu\nu}=-g_{\mu\nu}+\frac{p_\mu p_\nu}{W^2}, \tag{35}$$

$$\mathcal{P}^{(2)}_{\mu_1\mu_2;\nu_1\nu_2}=\frac{1}{2}(\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2}+\tilde{g}_{\mu_2\nu_1}\tilde{g}_{\mu_1\nu_2})-\frac{1}{3}\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\nu_1\nu_2}, \tag{36}$$

$$\begin{aligned}
\mathcal{P}^{(3)}_{\mu_1\mu_2\mu_3;\nu_1\nu_2\nu_3} &=\\
+\tfrac{1}{6}\sum_{P\{\nu_1,\nu_2,\nu_3\}} &\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2}\tilde{g}_{\mu_3\nu_3}\\
-\tfrac{1}{30}\sum_{P\{\nu_1,\nu_2,\nu_3\}} &(\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\nu_1\nu_2}\tilde{g}_{\mu_3\nu_3}+\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\mu_3}\tilde{g}_{\nu_2\nu_3}+\tilde{g}_{\mu_1\mu_3}\tilde{g}_{\nu_1\nu_3}\tilde{g}_{\mu_2\nu_2}),
\end{aligned} \tag{37}$$

$$\begin{aligned}
\mathcal{P}^{(4)}_{\mu_1\mu_2\mu_3\mu_4;\nu_1\nu_2\nu_3\nu_4} &=\\
+\tfrac{1}{24}\sum_{P\{\nu_1,\nu_2,\nu_3,\nu_4\}} &\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2}\tilde{g}_{\mu_3\nu_3}\tilde{g}_{\mu_4\nu_4}\\
-\tfrac{1}{168}\sum_{P\{\nu_1,\nu_2,\nu_3,\nu_4\}} &(\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\mu_3\nu_1}\tilde{g}_{\mu_4\nu_2}+\tilde{g}_{\mu_2\mu_3}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_4\nu_2}+\tilde{g}_{\mu_1\mu_3}\tilde{g}_{\mu_2\nu_1}\tilde{g}_{\mu_4\nu_2}+\\
&\tilde{g}_{\mu_1\mu_4}\tilde{g}_{\mu_2\nu_1}\tilde{g}_{\mu_3\nu_2}+\tilde{g}_{\mu_2\mu_4}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_3\nu_2}+\tilde{g}_{\mu_3\mu_4}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2})\tilde{g}_{\nu_3\nu_4}\\
+\tfrac{1}{840}\sum_{P\{\nu_1,\nu_2,\nu_3,\nu_4\}} &(\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\mu_3\mu_4}+\tilde{g}_{\mu_1\mu_3}\tilde{g}_{\mu_2\mu_4}+\tilde{g}_{\mu_1\mu_4}\tilde{g}_{\mu_2\mu_3})\tilde{g}_{\nu_1\nu_2}\tilde{g}_{\nu_3\nu_4},
\end{aligned} \tag{38}$$

$$
\begin{array}{lll}
\mathcal{P}^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5;\nu_1\nu_2\nu_3\nu_4\nu_5} & = & \\
+\frac{1}{120} & \sum\limits_{P\{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5\}} & \tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2}\tilde{g}_{\mu_3\nu_3}\tilde{g}_{\mu_4\nu_4}\tilde{g}_{\mu_5\nu_5} \\
-\frac{1}{1080} & \sum\limits_{P\{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5\}} & (\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\mu_3\nu_1}\tilde{g}_{\mu_4\nu_2}\tilde{g}_{\mu_5\nu_3} + \tilde{g}_{\mu_1\mu_3}\tilde{g}_{\mu_2\nu_1}\tilde{g}_{\mu_4\nu_2}\tilde{g}_{\mu_5\nu_3} \\
 & & +\tilde{g}_{\mu_1\mu_4}\tilde{g}_{\mu_2\nu_1}\tilde{g}_{\mu_3\nu_2}\tilde{g}_{\mu_5\nu_3} + \tilde{g}_{\mu_1\mu_5}\tilde{g}_{\mu_2\nu_1}\tilde{g}_{\mu_3\nu_2}\tilde{g}_{\mu_4\nu_3} \\
 & & +\tilde{g}_{\mu_2\mu_3}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_4\nu_2}\tilde{g}_{\mu_5\nu_3} + \tilde{g}_{\mu_2\mu_4}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_3\nu_2}\tilde{g}_{\mu_5\nu_3} \\
 & & +\tilde{g}_{\mu_2\mu_5}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_3\nu_2}\tilde{g}_{\mu_4\nu_3} + \tilde{g}_{\mu_3\mu_4}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2}\tilde{g}_{\mu_5\nu_3} \\
 & & +\tilde{g}_{\mu_3\mu_5}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2}\tilde{g}_{\mu_4\nu_3} + \tilde{g}_{\mu_4\mu_5}\tilde{g}_{\mu_1\nu_1}\tilde{g}_{\mu_2\nu_2}\tilde{g}_{\mu_3\nu_3})\tilde{g}_{\nu_4\nu_5} \\
+\frac{1}{7560} & \sum\limits_{P\{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5\}} & (\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\mu_3\mu_4}\tilde{g}_{\mu_5\nu_1} + \tilde{g}_{\mu_1\mu_3}\tilde{g}_{\mu_2\mu_4}\tilde{g}_{\mu_5\nu_1} + \tilde{g}_{\mu_1\mu_4}\tilde{g}_{\mu_2\mu_3}\tilde{g}_{\mu_5\nu_1} \\
 & & +\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\mu_3\mu_5}\tilde{g}_{\mu_4\nu_1} + \tilde{g}_{\mu_1\mu_3}\tilde{g}_{\mu_2\mu_5}\tilde{g}_{\mu_4\nu_1} + \tilde{g}_{\mu_1\mu_5}\tilde{g}_{\mu_2\mu_3}\tilde{g}_{\mu_4\nu_1} \\
 & & +\tilde{g}_{\mu_1\mu_2}\tilde{g}_{\mu_4\mu_5}\tilde{g}_{\mu_3\nu_1} + \tilde{g}_{\mu_1\mu_4}\tilde{g}_{\mu_2\mu_5}\tilde{g}_{\mu_3\nu_1} + \tilde{g}_{\mu_1\mu_5}\tilde{g}_{\mu_2\mu_4}\tilde{g}_{\mu_3\nu_1} \\
 & & +\tilde{g}_{\mu_1\mu_3}\tilde{g}_{\mu_4\mu_5}\tilde{g}_{\mu_2\nu_1} + \tilde{g}_{\mu_1\mu_4}\tilde{g}_{\mu_3\mu_5}\tilde{g}_{\mu_2\nu_1} + \tilde{g}_{\mu_1\mu_5}\tilde{g}_{\mu_3\mu_4}\tilde{g}_{\mu_2\nu_1} \\
 & & +\tilde{g}_{\mu_2\mu_3}\tilde{g}_{\mu_4\mu_5}\tilde{g}_{\mu_1\nu_1} + \tilde{g}_{\mu_2\mu_4}\tilde{g}_{\mu_3\mu_5}\tilde{g}_{\mu_1\nu_1} + \tilde{g}_{\mu_2\mu_5}\tilde{g}_{\mu_3\mu_4}\tilde{g}_{\mu_1\nu_1})\tilde{g}_{\nu_2\nu_3}\tilde{g}_{\nu_4\nu_5}.
\end{array}
\tag{39}
$$

Now we can state Feynman rules for bosons as: (1) Every incoming particle or incoming antiparticle contributes a factor of $e_{\mu_1\mu_2...\mu_j}(\vec{p},j,\sigma)$. (2) Every outgoing particle or outgoing antiparticle contributes a factor of $e^*_{\mu_1\mu_2...\mu_j}(\vec{p},j,\sigma)$. (3) For each spin-$j$ internal line, include a factor [24]

$$
i\frac{\mathcal{P}^{(j)}_{\mu_1\mu_2...\mu_j;\nu_1\nu_2...\nu_j}}{p^2-W^2+i\epsilon}. \tag{40}
$$

Note that we have dropped constants unnecessary for amplitude analysis. The denominator of Eq. (40) is often changed to Breit-Wigner form as an approximation to the full propagator:

$$
i\frac{\mathcal{P}^{(j)}_{\mu_1\mu_2...\mu_j;\nu_1\nu_2...\nu_j}}{p^2-W^2+i\Gamma W}, \tag{41}
$$

where Γ is the width of the particle. The width is either determined from experiments or from the chain approximation of theoretical models.

The problem left now is how to write down effective vertexes. We will give a list of 3-leg effective vertexes in next section.

III. THREE-LEG EFFECTIVE VERTEXES

The effective vertex Γ should be constructed from momentums and isotropic tensors of the Lorentz group. Free indexes left after contraction of $p_i^\mu, g^{\mu\nu}, \varepsilon^{\alpha\beta\gamma\delta}$ are the Lorentz indexes of Γ. Here p_i^μ $(i=1,2,...,n)$ is the four-momenta for particle i and $\varepsilon^{\alpha\beta\gamma\delta}$ the antisymmetric tensor. Not all possible constructions are independent since we have conservation of energy and momentum

$$
p_1^\mu + p_2^\mu + ... + p_n^\mu = 0 \tag{42}
$$

and Rarita-Schwinger conditions.

The antisymmetric tensor has the property

$$
\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'} = -\det\left\{\begin{array}{cccc}
g_{\mu\mu'} & g_{\nu\mu'} & g_{\alpha\mu'} & g_{\beta\mu'} \\
g_{\mu\nu'} & g_{\nu\nu'} & g_{\alpha\nu'} & g_{\beta\nu'} \\
g_{\mu\alpha'} & g_{\nu\alpha'} & g_{\alpha\alpha'} & g_{\beta\alpha'} \\
g_{\mu\beta'} & g_{\nu\beta'} & g_{\alpha\beta'} & g_{\beta\beta'}
\end{array}\right\}, \tag{43}
$$

so products of $\varepsilon_{\mu\nu\alpha\beta}$ can be absorbed into other terms K. S. freely.

Let's move on to the case of effective vertexes with only three legs.

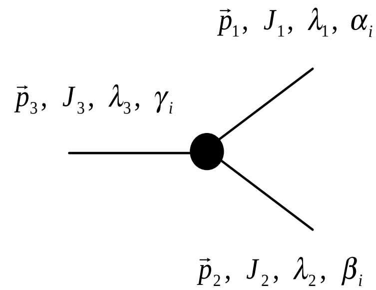

FIG. 1. A three-leg effective vertex

As shown in Fig. 1, the four-momenta of the three particles are p_1, p_2 and p_3. The spins are J_1, J_2 and J_3, and helicities being λ_1, λ_2 and λ_3. The Lorentz indexes for these particles are $(\alpha_1, \alpha_2, ..., \alpha_{J_1})$, $(\beta_1, \beta_2, ..., \beta_{J_2})$ and $(\gamma_1, \gamma_2, ..., \gamma_{J_3})$, so the effective vertex Γ has $J_1 + J_2 + J_3$ indexes.

The antisymmetric tensor $\varepsilon^{\alpha\beta\gamma\delta}$ in a three-leg vertex should contract with at least one four-momenta since wave functions are symmetric.

We define

$$A_1^{\alpha\beta\gamma} \equiv p_{1\mu}\varepsilon^{\mu\alpha\beta\gamma}, \tag{44}$$
$$A_2^{\alpha\beta\gamma} \equiv p_{2\mu}\varepsilon^{\mu\alpha\beta\gamma}, \tag{45}$$
$$Q^{\mu\nu} \equiv p_{1\alpha}p_{2\beta}\varepsilon^{\alpha\beta\mu\nu}. \tag{46}$$

In effective vertexes involving higher spin particles, some seemingly independent terms we write down are in fact not independent. There are equivalence relations among them. These equivalence relations come from Eq. (43) and the following identity [20]

$$\varepsilon^{\alpha\beta\gamma\delta}g^{\mu\nu} = \varepsilon^{\mu\beta\gamma\delta}g^{\alpha\nu} + \varepsilon^{\alpha\mu\gamma\delta}g^{\beta\nu} + \varepsilon^{\alpha\beta\mu\delta}g^{\gamma\nu} + \varepsilon^{\alpha\beta\gamma\mu}g^{\delta\nu}. \tag{47}$$

We can find[1]

$$Q^{\alpha_1\gamma_1}p_1^{\beta_1} \simeq p_1^2 A_2^{\alpha_1\beta_1\gamma_1} - (p_1\cdot p_2)A_1^{\alpha_1\beta_1\gamma_1} + Q^{\alpha_1\beta_1}p_1^{\gamma_1}, \tag{48}$$
$$Q^{\beta_1\gamma_1}p_2^{\alpha_1} \simeq -(p_1\cdot p_2)A_2^{\alpha_1\beta_1\gamma_1} + p_2^2 A_1^{\alpha_1\beta_1\gamma_1} + Q^{\alpha_1\beta_1}p_1^{\gamma_1}, \tag{49}$$

$$Q^{\alpha_1\beta_1}g^{\alpha_2\gamma_1} \simeq A_1^{\alpha_1\beta_1\gamma_1}p_2^{\alpha_2} + Q^{\alpha_1\gamma_1}g^{\alpha_2\beta_1}, \tag{50}$$
$$Q^{\alpha_1\beta_1}g^{\beta_2\gamma_1} \simeq -A_2^{\alpha_1\beta_1\gamma_1}p_1^{\beta_2} - Q^{\beta_1\gamma_1}g^{\alpha_1\beta_2}, \tag{51}$$

$$Q^{\alpha_1\gamma_1}g^{\beta_1\gamma_2} - Q^{\beta_1\gamma_1}g^{\alpha_1\gamma_2} \simeq A_1^{\alpha_1\beta_1\gamma_1}p_1^{\gamma_2} + A_2^{\alpha_1\beta_1\gamma_1}p_1^{\gamma_2}, \tag{52}$$

$$\begin{aligned} & g^{\alpha_1\gamma_1}p_1^{\beta_1}p_1^{\beta_2}p_1^{\gamma_2}p_2^{\alpha_2} - g^{\beta_1\gamma_1}p_1^{\beta_2}p_1^{\gamma_2}p_2^{\alpha_1}p_2^{\alpha_2} \\ \simeq\ & g^{\alpha_1\beta_1}p_1^{\beta_2}p_1^{\gamma_1}p_1^{\gamma_2}p_2^{\alpha_2} + [(p_1\cdot p_2)^2 - p_1^2p_2^2]g^{\alpha_1\beta_1}g^{\alpha_2\gamma_1}g^{\beta_2\gamma_2} \\ & -\tfrac{1}{2}p_2^2 g^{\alpha_1\gamma_1}g^{\alpha_2\gamma_2}p_1^{\beta_1}p_1^{\beta_2} - \tfrac{1}{2}p_1^2 g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}p_2^{\alpha_1}p_2^{\alpha_2} \\ & -\tfrac{1}{2}(p_1+p_2)^2 g^{\alpha_1\beta_1}g^{\alpha_2\beta_2}p_1^{\gamma_1}p_1^{\gamma_2} - (p_1\cdot p_2)g^{\alpha_1\gamma_1}g^{\beta_1\gamma_2}p_1^{\beta_2}p_2^{\alpha_2} \\ & +[(p_1\cdot p_2)+p_2^2]g^{\alpha_1\beta_1}g^{\alpha_2\gamma_1}p_1^{\beta_2}p_1^{\gamma_2} - [(p_1\cdot p_2)+p_1^2]g^{\alpha_1\beta_1}g^{\beta_2\gamma_1}p_1^{\gamma_2}p_2^{\alpha_2}, \end{aligned} \tag{53}$$

$$A_1^{\alpha_1\beta_1\gamma_1}p_2^{\alpha_2}g^{\beta_2\gamma_2} + A_2^{\alpha_1\beta_1\gamma_1}g^{\alpha_2\gamma_2}p_1^{\beta_2} \simeq -g^{\alpha_1\beta_1}Q^{\alpha_2\gamma_1}g^{\beta_2\gamma_2} - g^{\alpha_1\beta_1}Q^{\beta_2\gamma_1}g^{\alpha_2\gamma_2}, \tag{54}$$

[1] Three of these equivalence relations, Eq. (48), Eq. (49) and Eq. (52), have been listed in Ref. [20].

$$
\begin{array}{rl}
& A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_2} + A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_2} \\
\simeq & \frac{1}{2} p_1^2 A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} g^{\beta_2\gamma_2} - \frac{1}{2}(p_1\cdot p_2) A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} g^{\beta_2\gamma_2} \\
& +\frac{1}{2}(p_1\cdot p_2) A_2^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} p_1^{\beta_2} - \frac{1}{2} p_2^2 A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} p_1^{\beta_2} \\
& +\frac{1}{2}[(p_1\cdot p_2) - p_1^2] g^{\alpha_1\beta_1} A_2^{\alpha_2\beta_2\gamma_1} p_1^{\gamma_2} \\
& +\frac{1}{2}[(p_1\cdot p_2) - p_2^2] g^{\alpha_1\beta_1} A_1^{\alpha_2\beta_2\gamma_1} p_1^{\gamma_2} \\
& -g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} p_1^{\gamma_1} p_1^{\gamma_2}.
\end{array}
\tag{55}
$$

The symbol "$\simeq$" means that the left hand side and the right hand side of the equation are equal only when they are contracted with wave functions satisfying Eqs. (29-31). The left hand side of these equations can be absorbed into those terms on the right hand side without introducing kinematic singularities.

A K. S. free effective vertex is written as

$$
\Gamma = \Gamma_+ + \Gamma_-, \tag{56}
$$

where $\Gamma_+(\Gamma_-)$ is the tensor(pseudo-tensor) part of the vertex. We sort the three particles in ascending order of their spins, i.e., $J_1 \le J_2 \le J_3$. After considering these equivalence relations, one finds:

- $(J_1, J_2, J_3) = (0, 0, j)$, with $j \le 0$

$$
\Gamma_+ = c_1 \left(p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j} \right), \tag{57}
$$

$$
\Gamma_- = 0; \tag{58}
$$

- $(J_1, J_2, J_3) = (0, j, j^{'})$, with $1 \le j \le j^{'}$

$$
\begin{array}{rcl}
\Gamma_+ & = & c_1 \left(g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}} \cdots p_1^{\gamma_{j'}} \right) \\
& & +c_2 \left(g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_{j'}} \right) \\
& & +\cdots \\
& & +c_j \left(g^{\beta_1\gamma_1} p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j^{'}} \right) \\
& & +c_{j+1} \left(p_1^{\beta_1} p_1^{\beta_2} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j^{'}} \right),
\end{array}
\tag{59}
$$

$$
\begin{array}{rcl}
\Gamma_- & = & Q^{\beta_1\gamma_1} \left\{ c_{j+2} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}} \cdots p_1^{\gamma_j^{'}} \right) \right. \\
& & +c_{j+3} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_j^{'}} \right) \\
& & +\cdots \\
& & +c_{2j} \left(g^{\beta_2\gamma_2} p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_{j'}} \right) \\
& & \left. +c_{2j+1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j'}} \right) \right\};
\end{array}
\tag{60}
$$

- $(J_1, J_2, J_3) = (1, j, j)$, with $j \le 1$

$$
\begin{aligned}
\Gamma_+ \;=\;\; & g^{\alpha_1\gamma_1} && \Big\{c_1\left(g^{\beta_1\gamma_2}g^{\beta_2\gamma_3}\cdots g^{\beta_{j-1}\gamma_j}p_1^{\beta_j}\right)\\
& && +c_2\left(g^{\beta_1\gamma_2}g^{\beta_2\gamma_3}\cdots g^{\beta_{j-2}\gamma_{j-1}}p_1^{\beta_{j-1}}p_1^{\beta_j}p_1^{\gamma_j}\right)\\
& && +\cdots\\
& && +c_j\left(p_1^{\beta_1}p_1^{\beta_2}\cdots p_1^{\beta_j}p_1^{\gamma_2}p_1^{\gamma_3}\cdots p_1^{\gamma_j}\right)\Big\}\\
& +p_2^{\alpha_1} && \Big\{c_{j+1}\left(g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}\cdots g^{\beta_j\gamma_j}\right)\\
& && +c_{j+2}\left(g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}\cdots g^{\beta_{j-1}\gamma_{j-1}}p_1^{\beta_j}p_1^{\gamma_j}\right)\\
& && +\cdots\\
& && +c_{2j+1}\left(p_1^{\beta_1}p_1^{\beta_2}\cdots p_1^{\beta_j}p_1^{\gamma_1}p_1^{\gamma_2}\cdots p_1^{\gamma_j}\right)\Big\}\\
& +g^{\alpha_1\beta_1} && \Big\{c_{2j+2}\left(g^{\beta_2\gamma_1}g^{\beta_3\gamma_2}\cdots g^{\beta_j\gamma_{j-1}}p_1^{\gamma_j}\right)\\
& && +c_{2j+3}\left(g^{\beta_2\gamma_1}g^{\beta_3\gamma_2}\cdots g^{\beta_{j-1}\gamma_{j-2}}p_1^{\beta_j}p_1^{\gamma_{j-1}}p_1^{\gamma_j}\right)\\
& && +\cdots\\
& && +c_{3j+1}\left(p_1^{\beta_2}p_1^{\beta_3}\cdots p_1^{\beta_j}p_1^{\gamma_1}p_1^{\gamma_2}\cdots p_1^{\gamma_j}\right)\Big\},
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
\Gamma_- \;=\;\; & A_1^{\alpha_1\beta_1\gamma_1} && \Big\{c_{3j+2}\left(g^{\beta_2\gamma_2}g^{\beta_3\gamma_3}\cdots g^{\beta_j\gamma_j}\right)\\
& && +c_{3j+3}\left(g^{\beta_2\gamma_2}g^{\beta_3\gamma_3}\cdots g^{\beta_j\gamma_j}\right)\\
& && +\cdots\\
& && +c_{4j+1}\left(p_1^{\beta_2}p_1^{\beta_2}\cdots p_1^{\beta_j}p_1^{\gamma_2}p_1^{\gamma_3}\cdots p_1^{\gamma_j}\right)\Big\}\\
& +A_2^{\alpha_1\beta_1\gamma_1} && \Big\{c_{4j+2}\left(g^{\beta_2\gamma_2}g^{\beta_3\gamma_3}\cdots g^{\beta_j\gamma_j}\right)\\
& && +c_{4j+3}\left(g^{\beta_2\gamma_2}g^{\beta_3\gamma_3}\cdots g^{\beta_j\gamma_j}\right)\\
& && +\cdots\\
& && +c_{5j+1}\left(p_1^{\beta_2}p_1^{\beta_2}\cdots p_1^{\beta_j}p_1^{\gamma_2}p_1^{\gamma_3}\cdots p_1^{\gamma_j}\right)\Big\}\\
& +Q^{\alpha_1\beta_1} && \Big\{c_{5j+2}\left(g^{\beta_2\gamma_1}g^{\beta_3\gamma_2}\cdots g^{\beta_j\gamma_{j-1}}p_1^{\gamma_j}\right)\\
& && +c_{5j+3}\left(g^{\beta_2\gamma_1}g^{\beta_3\gamma_2}\cdots g^{\beta_{j-1}\gamma_{j-2}}p_1^{\beta_j}p_1^{\gamma_{j-1}}p_1^{\gamma_j}\right)\\
& && +\cdots\\
& && +c_{6j+1}\left(p_1^{\beta_2}p_1^{\beta_3}\cdots p_1^{\beta_j}p_1^{\gamma_1}p_1^{\gamma_2}\cdots p_1^{\gamma_j}\right)\Big\};
\end{aligned}
\tag{62}
$$

- $(J_1, J_2, J_3) = (1, j, j^{'})$, with $1 \leq j < j^{'}$

$$
\begin{aligned}
\Gamma_+ \;=\;\; & g^{\alpha_1\gamma_1} && \Big\{c_1\left(g^{\beta_1\gamma_2}g^{\beta_2\gamma_3}\cdots g^{\beta_j\gamma_{j+1}}p_1^{\gamma_{j+2}}p_1^{\gamma_{j+3}}\cdots p_1^{\gamma_{j'}}\right)\\
& && +c_2\left(g^{\beta_1\gamma_2}g^{\beta_2\gamma_3}\cdots g^{\beta_{j-1}\gamma_j}p_1^{\beta_j}p_1^{\gamma_{j+1}}p_1^{\gamma_{j+2}}\cdots p_1^{\gamma_{j'}}\right)\\
& && +\cdots\\
& && +\ c_{j+1}\left(p_1^{\beta_1}p_1^{\beta_2}\cdots p_1^{\beta_j}p_1^{\gamma_2}p_1^{\gamma_3}\cdots p_1^{\gamma_j'}\right)\Big\}\\
& +p_2^{\alpha_1} && \Big\{c_{j+1}\left(g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}\cdots g^{\beta_j\gamma_j}p_1^{\gamma_{j+1}}p_1^{\gamma_{j+2}}\cdots p_1^{\gamma_{j'}}\right)\\
& && +c_{j+2}\left(g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}\cdots g^{\beta_{j-1}\gamma_{j-1}}p_1^{\beta_j}p_1^{\gamma_j}p_1^{\gamma_{j+1}}\cdots p_1^{\gamma_{j'}}\right)\\
& && +\cdots\\
& && +\ c_{2j+2}\left(p_1^{\beta_1}p_1^{\beta_2}\cdots p_1^{\beta_j}p_1^{\gamma_1}p_1^{\gamma_2}\cdots p_1^{\gamma_j'}\right)\Big\}\\
& +g^{\alpha_1\beta_1} && \Big\{c_{2j+3}\left(g^{\beta_2\gamma_1}g^{\beta_3\gamma_2}\cdots g^{\beta_j\gamma_{j-1}}p_1^{\gamma_j}p_1^{\gamma_{j+1}}\cdots p_1^{\gamma_{j'}}\right)\\
& && +c_{2j+4}\left(g^{\beta_2\gamma_1}g^{\beta_3\gamma_2}\cdots g^{\beta_{j-1}\gamma_{j-2}}p_1^{\beta_j}p_1^{\gamma_{j-1}}p_1^{\gamma_j}\cdots p_1^{\gamma_{j'}}\right)\\
& && +\cdots\\
& && +c_{3j+2}\left(p_1^{\beta_2}p_1^{\beta_3}\cdots p_1^{\beta_j}p_1^{\gamma_1}p_1^{\gamma_2}\cdots p_1^{\gamma_{j'}}\right)\Big\},
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
\Gamma_- \quad = \quad & A_1^{\alpha_1\beta_1\gamma_1} && \Big\{ c_{3j+3} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}} \cdots p_1^{\gamma_{j'}} \right) \\
& && + c_{3j+4} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_{j'}} \right) \\
& && + \cdots \\
& && + c_{4j+2} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j'} \right) \Big\} \\
& + A_2^{\alpha_1\beta_1\gamma_1} && \Big\{ c_{4j+3} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}} \cdots p_1^{\gamma_{j'}} \right) \\
& && + c_{4j+4} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_{j'}} \right) \\
& && + \cdots \\
& && + c_{5j+2} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j'} \right) \Big\} \\
& + Q^{\alpha_1\gamma_1} && c_{5j+3} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3} \cdots g^{\beta_j\gamma_{j+1}} p_1^{\gamma_{j+2}} p_1^{\gamma_{j+3}} \cdots p_1^{\gamma_{j'}} \right) \\
& + Q^{\alpha_1\beta_1} && \Big\{ c_{5j+4} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_{j'}} \right) \\
& && + c_{5j+5} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \cdots p_1^{\gamma_{j'}} \right) \\
& && + c_{6j+3} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j'}} \right) \Big\} ;
\end{aligned}
\tag{64}
$$

- $(J_1, J_2, J_3) = (2, 2, 2)$

$$
\begin{aligned}
\Gamma_+ \quad = \quad & c_1 g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} \\
& + c_2 g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} g^{\beta_2\gamma_2} \\
& + c_3 g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} p_1^{\beta_2} p_1^{\gamma_2} \\
& + c_4 g^{\alpha_1\beta_1} p_2^{\alpha_2} g^{\beta_2\gamma_1} p_1^{\gamma_2} \\
& + c_5 g^{\alpha_1\beta_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} \\
& + c_6 g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} p_1^{\beta_1} p_1^{\beta_2} \\
& + c_7 g^{\alpha_1\gamma_1} p_2^{\alpha_2} g^{\beta_1\gamma_2} p_1^{\beta_2} \\
& + c_8 g^{\alpha_1\gamma_1} p_2^{\alpha_2} p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_2} \\
& + c_9 p_2^{\alpha_1} p_2^{\alpha_2} g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} \\
& + c_{10} p_2^{\alpha_1} p_2^{\alpha_2} p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} ,
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
\Gamma_- \quad = \quad & c_{11} g^{\alpha_1\beta_1} A_1^{\alpha_2\beta_2\gamma_1} p_1^{\gamma_2} \\
& + c_{12} g^{\alpha_1\beta_1} A_2^{\alpha_2\beta_2\gamma_1} p_1^{\gamma_2} \\
& + c_{13} g^{\alpha_1\beta_1} Q^{\alpha_2\gamma_1} g^{\beta_2\gamma_2} \\
& + c_{14} g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} \\
& + c_{15} A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} p_1^{\beta_2} \\
& + c_{16} A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} g^{\beta_2\gamma_2} \\
& + c_{17} A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_2} \\
& + c_{18} A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} g^{\beta_2\gamma_2} \\
& + c_{19} Q^{\alpha_1\beta_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} ;
\end{aligned}
\tag{66}
$$

- $(J_1, J_2, J_3) = (2, j, j)$, with $j \geq 3$

$$
\begin{aligned}
\Gamma_+ \;=\; & g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} && \left\{c_1 \left(g^{\beta_3\gamma_1} g^{\beta_4\gamma_2} \cdots g^{\beta_j\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j}\right)\right. \\
& && +c_2 \left(g^{\beta_3\gamma_1} g^{\beta_4\gamma_2} \cdots g^{\beta_{j-1}\gamma_{j-3}} p_1^{\beta_j} p_1^{\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j}\right) \\
& && +\cdots \\
& && \left. +c_{j-1} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j}\right)\right\} \\
& +g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} && \left\{c_j \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j}\right)\right. \\
& && +c_{j+1} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j}\right) \\
& && +\cdots \\
& && \left. +c_{2j-1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j}\right)\right\} \\
& +g^{\alpha_1\beta_1} p_2^{\alpha_2} && \left\{c_{2j} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j}\right)\right. \\
& && +c_{2j+1} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j}\right) \\
& && +\cdots \\
& && \left. +c_{3j-1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j}\right)\right\} \\
& +g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} && \left\{c_{3j} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} \cdots g^{\beta_{j-2}\gamma_j} p_1^{\beta_{j-1}} p_1^{\beta_j}\right)\right. \\
& && +c_{3j+1} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} \cdots g^{\beta_{j-3}\gamma_{j-1}} p_1^{\beta_{j-2}} p_1^{\beta_{j-1}} p_1^{\beta_j} p_1^{\gamma_j}\right) \\
& && +\cdots \\
& && \left. +c_{4j-2} \left(p_1^{\beta_1} p_1^{\beta_2} \cdots p_1^{\beta_j} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_j}\right)\right\} \\
& +g^{\alpha_1\gamma_1} p_2^{\alpha_2} && \left\{c_{4j-1} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3} \cdots g^{\beta_{j-1}\gamma_j} p_1^{\beta_j}\right)\right. \\
& && +c_{4j} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3} \cdots g^{\beta_{j-1}\gamma_j} p_1^{\beta_j}\right) \\
& && +\cdots \\
& && \left. +c_{5j-2} \left(p_1^{\beta_1} p_1^{\beta_2} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j}\right)\right\} \\
& +p_2^{\alpha_1} p_2^{\alpha_2} && c_{5j-1} \left(g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} \cdots g^{\beta_j\gamma_j}\right) \\
& +p_2^{\alpha_1} p_2^{\alpha_2} && c_{5j} \left(p_1^{\beta_1} p_1^{\beta_2} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j}\right),
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\Gamma_- \;=\;& g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} Q^{\beta_3\gamma_1} && \Big\{ c_{5j+1} \left(g^{\beta_4\gamma_2} g^{\beta_5\gamma_3} \cdots g^{\beta_j\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \right) \\
&&& + c_{5j+2} \left(g^{\beta_4\gamma_2} g^{\beta_5\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-3}} p_1^{\beta_j} p_1^{\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \right) \\
&&& + \cdots \\
&&& + c_{6j-2} \left(p_1^{\beta_4} p_1^{\beta_5} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} \right) \Big\} \\
& + g^{\alpha_1\beta_1} A_1^{\alpha_2\beta_2\gamma_1} && \Big\{ c_{6j-1} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} \right) \\
&&& + c_{6j} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \right) \\
&&& + \cdots \\
&&& + c_{7j-3} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} \right) \Big\} \\
& + g^{\alpha_1\beta_1} A_2^{\alpha_2\beta_2\gamma_1} && \Big\{ c_{7j-2} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} \right) \\
&&& + c_{7j-1} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \right) \\
&&& + \cdots \\
&&& + c_{8j-4} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} \right) \Big\} \\
& + g^{\alpha_1\beta_1} Q^{\alpha_2\gamma_1} && c_{8j-3} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} \right) \\
& + g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} && c_{8j-2} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j} \right) \\
& + A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} && \Big\{ c_{8j-1} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_{j-1}\gamma_j} p_1^{\beta_j} \right) \\
&&& + c_{8j} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_{j-2}\gamma_{j-1}} p_1^{\beta_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} \right) \\
&&& + \cdots \\
&&& + c_{9j-3} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_j} \right) \Big\} \\
& + A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} && \Big\{ c_{9j-2} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} \right) \\
&&& + c_{9j-1} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} \right) \\
&&& + \cdots \\
&&& + c_{10j-3} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} \right) \Big\} \\
& + A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} && c_{10j-2} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} \right) \\
& + Q^{\alpha_1\beta_1} p_2^{\alpha_2} && c_{10j-1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j} \right);
\end{aligned}
\tag{68}
$$

- $(J_1, J_2, J_3) = (2, 2, 3)$

$$
\begin{aligned}
\Gamma_+ \;=\;& c_1 g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_2 g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} g^{\beta_2\gamma_2} p_1^{\gamma_3} \\
& + c_3 g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} p_1^{\beta_2} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_4 g^{\alpha_1\beta_1} p_2^{\alpha_2} g^{\beta_2\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_5 g^{\alpha_1\beta_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_6 g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} g^{\beta_1\gamma_3} p_1^{\beta_2} \\
& + c_7 g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_3} \\
& + c_8 g^{\alpha_1\gamma_1} p_2^{\alpha_2} g^{\beta_1\gamma_2} g^{\beta_2\gamma_3} \\
& + c_9 g^{\alpha_1\gamma_1} p_2^{\alpha_2} g^{\beta_1\gamma_2} p_1^{\beta_2} p_1^{\gamma_3} \\
& + c_{10} g^{\alpha_1\gamma_1} p_2^{\alpha_2} p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_{11} p_2^{\alpha_1} p_2^{\alpha_2} g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} p_1^{\gamma_3} \\
& + c_{12} p_2^{\alpha_1} p_2^{\alpha_2} p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3},
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
\Gamma_- \;=\;\; & c_{13} g^{\alpha_1\beta_1} A_1^{\alpha_2\beta_2\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_{14} g^{\alpha_1\beta_1} A_2^{\alpha_2\beta_2\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_{15} g^{\alpha_1\beta_1} Q^{\alpha_2\gamma_1} g^{\beta_2\gamma_2} p^{\gamma_3} \\
& + c_{16} g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_{17} A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} g^{\beta_2\gamma_3} \\
& + c_{18} A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} p_1^{\beta_2} p_1^{\gamma_3} \\
& + c_{19} A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} g^{\beta_2\gamma_2} p_1^{\gamma_3} \\
& + c_{20} A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_2} p_1^{\gamma_3} \\
& + c_{21} A_2^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} g^{\beta_2\gamma_3} \\
& + c_{22} A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} g^{\beta_2\gamma_2} p_1^{\gamma_3} \\
& + c_{23} Q^{\alpha_1\beta_1} p_2^{\alpha_2} p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3};
\end{aligned}
\tag{70}
$$

- $(J_1, J_2, J_3) = (2, j, j+1)$, with $j \geq 3$

$$
\begin{aligned}
\Gamma_+ \;=\;\; & g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \; \Big\{ c_1 \left(g^{\beta_3\gamma_1} g^{\beta_4\gamma_2} \cdots g^{\beta_j\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + c_2 \left(g^{\beta_3\gamma_1} g^{\beta_4\gamma_2} \cdots g^{\beta_{j-1}\gamma_{j-3}} p_1^{\beta_j} p_1^{\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + \cdots \\
& \quad + c_{j-1} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} \; \Big\{ c_j \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + c_{j+1} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + \cdots \\
& \quad + c_{2j-1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \Big\} \\
& + g^{\alpha_1\beta_1} p_2^{\alpha_2} \; \Big\{ c_{2j} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + c_{2j+1} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + \cdots \\
& \quad + c_{3j-1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} \; \Big\{ c_{3j} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} \cdots g^{\beta_{j-1}\gamma_{j+1}} p_1^{\beta_j} \right) \\
& \quad + c_{3j+1} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} \cdots g^{\beta_{j-2}\gamma_j} p_1^{\beta_{j-1}} p_1^{\beta_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + \cdots \\
& \quad + c_{4j-1} \left(p_1^{\beta_1} p_1^{\beta_2} \cdots p_1^{\beta_j} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + g^{\alpha_1\gamma_1} p_2^{\alpha_2} \; \Big\{ c_{4j} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3} \cdots g^{\beta_j\gamma_{j+1}} \right) \\
& \quad + c_{4j+1} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3} \cdots g^{\beta_{j-1}\gamma_j} p_1^{\beta_j} p_1^{\gamma_{j+1}} \right) \\
& \quad + \cdots \\
& \quad + c_{5j} \left(p_1^{\beta_1} p_1^{\beta_2} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + p_2^{\alpha_1} p_2^{\alpha_2} \; c_{5j+1} \left(g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& + p_2^{\alpha_1} p_2^{\alpha_2} \; c_{5j+2} \left(p_1^{\beta_1} p_1^{\beta_2} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j+1}} \right),
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
\Gamma_- \;=\; & g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} Q^{\beta_3\gamma_1} && \Big\{ c_{5j+3} \left(g^{\beta_4\gamma_2} g^{\beta_5\gamma_3} \cdots g^{\beta_j\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + c_{5j+4} \left(g^{\beta_4\gamma_2} g^{\beta_5\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-3}} p_1^{\beta_j} p_1^{\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + \cdots \\
& && + c_{6j} \left(p_1^{\beta_4} p_1^{\beta_5} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + g^{\alpha_1\beta_1} A_1^{\alpha_2\beta_2\gamma_1} && \Big\{ c_{6j+1} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + c_{6j+2} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + \cdots \\
& && + c_{7j-1} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + g^{\alpha_1\beta_1} A_2^{\alpha_2\beta_2\gamma_1} && \Big\{ c_{7j} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + c_{7j+1} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + \cdots \\
& && + c_{8j-2} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + g^{\alpha_1\beta_1} Q^{\alpha_2\gamma_1} && c_{8j-1} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p^{\gamma_{j+1}} \right) \\
& + g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} && c_{8j} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j+1}} \right) \\
& + A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} && \Big\{ c_{8j+1} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_j\gamma_{j+1}} \right) \\
& && + c_{8j+2} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_{j-1}\gamma_j} p_1^{\beta_j} p_1^{\gamma_{j+1}} \right) \\
& && + \cdots \\
& && + c_{9j} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} && \Big\{ c_{9j+1} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + c_{9j+2} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& && + \cdots \\
& && + c_{10j} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j+1}} \right) \Big\} \\
& + A_2^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} && c_{10j+1} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_j\gamma_{j+1}} \right) \\
& + A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} && c_{10j+2} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} \right) \\
& + Q^{\alpha_1\beta_1} p_2^{\alpha_2} && c_{10j+3} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j+1}} \right);
\end{aligned}
\tag{72}
$$

- $(J_1, J_2, J_3) = (2, 2, j)$, with $j \geq 4$

$$
\begin{aligned}
\Gamma_+ \;=\; & c_1 g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \left(p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j} \right) \\
& + c_2 g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} \left(g^{\beta_2\gamma_2} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_j} \right) \\
& + c_3 g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} \left(p_1^{\beta_2} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} \right) \\
& + c_4 g^{\alpha_1\beta_1} p_2^{\alpha_2} \left(g^{\beta_2\gamma_1} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} \right) \\
& + c_5 g^{\alpha_1\beta_1} p_2^{\alpha_2} \left(p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j} \right) \\
& + c_6 g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} p_1^{\gamma_5} p_1^{\gamma_6} \cdots p_1^{\gamma_j} \right) \\
& + c_7 g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} \left(g^{\beta_1\gamma_3} p_1^{\beta_2} p_1^{\gamma_4} p_1^{\gamma_5} \cdots p_1^{\gamma_j} \right) \\
& + c_8 g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} \left(p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_j} \right) \\
& + c_9 g^{\alpha_1\gamma_1} p_2^{\alpha_2} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3} p_1^{\gamma_4} p_1^{\gamma_5} \cdots p_1^{\gamma_j} \right) \\
& + c_{10} g^{\alpha_1\gamma_1} p_2^{\alpha_2} \left(g^{\beta_1\gamma_2} p_1^{\beta_2} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_j} \right) \\
& + c_{11} g^{\alpha_1\gamma_1} p_2^{\alpha_2} \left(p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_j} \right) \\
& + c_{12} p_2^{\alpha_1} p_2^{\alpha_2} \left(g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_j} \right) \\
& + c_{13} p_2^{\alpha_1} p_2^{\alpha_2} \left(p_1^{\beta_1} p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_j} \right),
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
\Gamma_- \;=\;\; & c_{14} g^{\alpha_1\beta_1} A_1^{\alpha_2\beta_2\gamma_1}\left(p_1^{\gamma_2} p_1^{\gamma_3}\cdots p_1^{\gamma_j}\right) \\
& + c_{15} g^{\alpha_1\beta_1} A_2^{\alpha_2\beta_2\gamma_1}\left(p_1^{\gamma_2} p_1^{\gamma_3}\cdots p_1^{\gamma_j}\right) \\
& + c_{16} g^{\alpha_1\beta_1} Q^{\alpha_2\gamma_1}\left(g^{\beta_2\gamma_2} p^{\gamma_3} p^{\gamma_4}\cdots p^{\gamma_j}\right) \\
& + c_{17} g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2}\left(p_1^{\gamma_1} p_1^{\gamma_2}\cdots p_1^{\gamma_j}\right) \\
& + c_{18} A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2}\left(g^{\beta_2\gamma_3} p_1^{\gamma_4} p_1^{\gamma_5}\cdots p_1^{\gamma_j}\right) \\
& + c_{19} A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2}\left(p_1^{\beta_2} p_1^{\gamma_3} p_1^{\gamma_4}\cdots p_1^{\gamma_j}\right) \\
& + c_{20} A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2}\left(g^{\beta_2\gamma_2} p_1^{\gamma_3} p_1^{\gamma_4}\cdots p_1^{\gamma_j}\right) \\
& + c_{21} A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2}\left(p_1^{\beta_2} p_1^{\gamma_2} p_1^{\gamma_3}\cdots p_1^{\gamma_j}\right) \\
& + c_{22} A_2^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2}\left(g^{\beta_2\gamma_3} p_1^{\gamma_4} p_1^{\gamma_5}\cdots p_1^{\gamma_j}\right) \\
& + c_{23} A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2}\left(g^{\beta_2\gamma_2} p_1^{\gamma_3} p_1^{\gamma_4}\cdots p_1^{\gamma_j}\right) \\
& + c_{24} Q^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2}\left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} p_1^{\gamma_5} p_1^{\gamma_6}\cdots p_1^{\gamma_j}\right) \\
& + c_{25} Q^{\alpha_1\beta_1} p_2^{\alpha_2}\left(p_1^{\beta_2} p_1^{\gamma_1} p_1^{\gamma_2}\cdots p_1^{\gamma_j}\right);
\end{aligned}
\tag{74}
$$

- $(J_1, J_2, J_3) = (2, j, j^{'})$, with $j \geq 3$ and $j^{'} \geq j+2$

$$
\begin{aligned}
\Gamma_+ \;=\;\; & g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \;\Big\{ c_1 \left(g^{\beta_3\gamma_1} g^{\beta_4\gamma_2}\cdots g^{\beta_j\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + c_2 \left(g^{\beta_3\gamma_1} g^{\beta_4\gamma_2}\cdots g^{\beta_{j-1}\gamma_{j-3}} p_1^{\beta_j} p_1^{\gamma_{j-2}} p_1^{\gamma_{j-1}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + \cdots \\
& \qquad + c_{j-1} \left(p_1^{\beta_3} p_1^{\beta_4}\cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2}\cdots p_1^{\gamma_{j'}}\right)\Big\} \\
& + g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} \;\Big\{ c_j \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3}\cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + c_{j+1} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3}\cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + \cdots \\
& \qquad + c_{2j-1} \left(p_1^{\beta_2} p_1^{\beta_3}\cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3}\cdots p_1^{\gamma_{j'}}\right)\Big\} \\
& + g^{\alpha_1\beta_1} p_2^{\alpha_2} \;\Big\{ c_{2j} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2}\cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + c_{2j+1} \left(g^{\beta_2\gamma_1} g^{\beta_3\gamma_2}\cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + \cdots \\
& \qquad + c_{3j-1} \left(p_1^{\beta_2} p_1^{\beta_3}\cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2}\cdots p_1^{\gamma_{j'}}\right)\Big\} \\
& + g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} \;\Big\{ c_{3j} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4}\cdots g^{\beta_j\gamma_{j+2}} p_1^{\gamma_{j+3}} p_1^{\gamma_{j+4}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + c_{3j+1} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4}\cdots g^{\beta_{j-1}\gamma_{j+1}} p_1^{\beta_j} p_1^{\gamma_{j+2}} p_1^{\gamma_{j+3}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + \cdots \\
& \qquad + c_{4j} \left(p_1^{\beta_1} p_1^{\beta_2}\cdots p_1^{\beta_j} p_1^{\gamma_3} p_1^{\gamma_4}\cdots p_1^{\gamma_{j'}}\right)\Big\} \\
& + g^{\alpha_1\gamma_1} p_2^{\alpha_2} \;\Big\{ c_{4j+1} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3}\cdots g^{\beta_j\gamma_{j+1}} p_1^{\gamma_{j+2}} p_1^{\gamma_{j+3}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + c_{4j+2} \left(g^{\beta_1\gamma_2} g^{\beta_2\gamma_3}\cdots g^{\beta_{j-1}\gamma_j} p_1^{\beta_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}}\cdots p_1^{\gamma_{j'}}\right) \\
& \qquad + \cdots \\
& \qquad + c_{5j+1} \left(p_1^{\beta_1} p_1^{\beta_2}\cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3}\cdots p_1^{\gamma_{j'}}\right)\Big\} \\
& + p_2^{\alpha_1} p_2^{\alpha_2} \;\; c_{5j+2} \left(g^{\beta_1\gamma_1} g^{\beta_2\gamma_2}\cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}}\cdots p_1^{\gamma_{j'}}\right) \\
& + p_2^{\alpha_1} p_2^{\alpha_2} \;\; c_{5j+3} \left(p_1^{\beta_1} p_1^{\beta_2}\cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2}\cdots p_1^{\gamma_{j'}}\right),
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
\Gamma_- \;=\; g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} Q^{\beta_3\gamma_1} \;& \Big\{ c_{5j+4} \left(g^{\beta_4\gamma_2} g^{\beta_5\gamma_3} \cdots g^{\beta_j\gamma_{j-2}} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \cdots p_1^{\gamma_{j'}} \right) \\
& + c_{5j+5} \left(g^{\beta_4\gamma_2} g^{\beta_5\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-3}} p_1^{\beta_j} p_1^{\gamma_{j-2}} p_1^{\gamma_{j-1}} \cdots p_1^{\gamma_{j'}} \right) \\
& + \cdots \\
& + c_{6j+1} \left(p_1^{\beta_4} p_1^{\beta_5} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j'}} \right) \Big\} \\
+ g^{\alpha_1\beta_1} A_1^{\alpha_2\beta_2\gamma_1} \;& \Big\{ c_{6j+2} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_{j'}} \right) \\
& + c_{6j+3} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \cdots p_1^{\gamma_{j'}} \right) \\
& + \cdots \\
& + c_{7j} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j'}} \right) \Big\} \\
+ g^{\alpha_1\beta_1} A_2^{\alpha_2\beta_2\gamma_1} \;& \Big\{ c_{7j+1} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_j\gamma_{j-1}} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_{j'}} \right) \\
& + c_{7j+2} \left(g^{\beta_3\gamma_2} g^{\beta_4\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-2}} p_1^{\beta_j} p_1^{\gamma_{j-1}} p_1^{\gamma_j} \cdots p_1^{\gamma_{j'}} \right) \\
& + \cdots \\
& + c_{8j-1} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j'}} \right) \Big\} \\
+ g^{\alpha_1\beta_1} Q^{\alpha_2\gamma_1} \;& c_{8j} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p^{\gamma_{j+1}} p^{\gamma_{j+2}} \cdots p^{\gamma_{j'}} \right) \\
+ g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} \;& c_{8j+1} \left(p_1^{\beta_3} p_1^{\beta_4} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j'}} \right) \\
+ A_1^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} \;& \Big\{ c_{8j+2} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_j\gamma_{j+1}} p_1^{\gamma_{j+2}} p_1^{\gamma_{j+3}} \cdots p_1^{\gamma_{j'}} \right) \\
& + c_{8j+3} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_{j-1}\gamma_j} p_1^{\beta_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}} \cdots p_1^{\gamma_{j'}} \right) \\
& + \cdots \\
& + c_{9j+1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_3} p_1^{\gamma_4} \cdots p_1^{\gamma_{j'}} \right) \Big\} \\
+ A_1^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} \;& \Big\{ c_{9j+2} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}} \cdots p_1^{\gamma_{j'}} \right) \\
& + c_{9j+3} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_{j-1}\gamma_{j-1}} p_1^{\beta_j} p_1^{\gamma_j} p_1^{\gamma_{j+1}} \cdots p_1^{\gamma_{j'}} \right) \\
& + \cdots \\
& + c_{10j+1} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_2} p_1^{\gamma_3} \cdots p_1^{\gamma_{j'}} \right) \Big\} \\
+ A_2^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} \;& c_{10j+2} \left(g^{\beta_2\gamma_3} g^{\beta_3\gamma_4} \cdots g^{\beta_j\gamma_{j+1}} p_1^{\gamma_{j+2}} p_1^{\gamma_{j+3}} \cdots p_1^{\gamma_{j'}} \right) \\
+ A_2^{\alpha_1\beta_1\gamma_1} p_2^{\alpha_2} \;& c_{10j+3} \left(g^{\beta_2\gamma_2} g^{\beta_3\gamma_3} \cdots g^{\beta_j\gamma_j} p_1^{\gamma_{j+1}} p_1^{\gamma_{j+2}} \cdots p_1^{\gamma_{j'}} \right) \\
+ Q^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} \;& c_{10j+4} \left(g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} \cdots g^{\beta_j\gamma_{j+2}} p_1^{\gamma_{j+3}} p_1^{\gamma_{j+4}} \cdots p_1^{\gamma_{j'}} \right) \\
+ Q^{\alpha_1\beta_1} p_2^{\alpha_2} \;& c_{10j+5} \left(p_1^{\beta_2} p_1^{\beta_3} \cdots p_1^{\beta_j} p_1^{\gamma_1} p_1^{\gamma_2} \cdots p_1^{\gamma_{j'}} \right);
\end{aligned} \tag{76}
$$

etc. One can give more effective vertexes for those cases in which all three spins ≥ 3, but this will take too much space and these vertexes are seldom used. We drop the factor $\delta(p_1 + p_2 + p_3)$ in effective vertexes and amplitudes through out this paper. Scalar coefficients c_i are complex functions of the form

$$
c_i = c_i(p_1^2, p_2^2, p_1 \cdot p_2). \tag{77}
$$

They are just constants when all of the particles are on shell. These coefficients are also called invariant amplitudes, coupling coefficients, structure functions or form factors in different references.

The propagator of an off shell particle does not satisfy Eq. (29) and Eq. (31). In this case, those terms proportional to

$$
p_1^{\alpha_i},\ p_2^{\beta_i},\ (p_1 + p_2)^{\gamma_i},\ g^{\alpha_i\alpha_j},\ g^{\beta_i\beta_j},\ g^{\gamma_i\gamma_j} \tag{78}
$$

will not vanish. However, such an off shell term always contributes a factor of $p_1^2 - W_1^2$, $p_2^2 - W_2^2$ or $p_3^2 - W_3^2$ after contraction with the propagator. This factor will eliminate the denominator of the propagator, which makes the amplitude free of pole at the mass of the intermediate particle[2]. These contributions are not independent from

---

[2]See Ref. [24] for arguments on polology.

background terms. They can be absorbed into back ground amplitudes without introducing kinematic singularities. Off shell terms are not needed when back ground amplitudes are included. One can refer to Sec. VII or Ref. [22] for examples.

IV. SYMMETRY UNDER SPACE REFLECTION

Under space reflection operation $\mathbf{P}$, canonical quantum states transform as

$$\mathbf{P}|\vec{p}, j, m\rangle = \eta| - \vec{p}, j, m\rangle. \tag{79}$$

If parity is conserved

$$\mathbf{P}^{\dagger}\mathbf{S}\mathbf{P} = \mathbf{S}, \tag{80}$$

$$\begin{aligned}
&\langle\vec{p}_1^{\,'}, J_1^{'}, m_1^{'}; \vec{p}_2^{\,'}, J_2^{'}, m_2^{'}; \cdots|\mathbf{S}|\vec{p}_1, J_1, m_1; \vec{p}_2, J_2, m_2; \cdots\rangle \\
= \;& \eta_1^{'*}\eta_2^{'*}\cdots\eta_1\eta_2\cdots\langle-\vec{p}_1^{\,'}, J_1^{'}, m_1^{'}; -\vec{p}_2^{\,'}, J_2^{'}, m_2^{'}; \cdots|\mathbf{S}| - \vec{p}_1, J_1, m_1; -\vec{p}_2, J_2, m_2; \cdots\rangle,
\end{aligned} \tag{81}$$

where $\mathbf{S}$ is the S-matrix operator.

The space reflection matrix is defined as [9]

$$(P^{\mu}{}_{\nu}) = \mathrm{diag}\{1, -1, -1, -1\}, \tag{82}$$

we have

$$\bar{p}_i^{\mu} \equiv P^{\mu}{}_{\nu}p_i^{\nu} = (E_i, -\vec{p}_i)\,, \tag{83}$$

$$\bar{p}_i^{'\mu} \equiv P^{\mu}{}_{\nu}p_i^{'\nu} = \left(E_i^{'}, -\vec{p}_i^{\,'}\right), \tag{84}$$

$$\bar{g}^{\mu\nu} \equiv P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}g^{\alpha\beta} = g^{\mu\nu}, \tag{85}$$

$$\bar{\varepsilon}^{\alpha\beta\gamma\delta} \equiv P^{\alpha}{}_{\alpha'}P^{\beta}{}_{\beta'}P^{\gamma}{}_{\gamma'}P^{\delta}{}_{\delta'}\varepsilon^{\alpha'\beta'\gamma'\delta'} = -\varepsilon^{\alpha\beta\gamma\delta}, \tag{86}$$

$$\bar{e}^{\mu_1\mu_2\cdots\mu_j} \equiv P^{\mu_1}{}_{\nu_1}P^{\mu_2}{}_{\nu_2}\cdots P^{\mu_j}{}_{\nu_j}e^{\nu_1\nu_2\cdots\nu_j}. \tag{87}$$

Eq. (81) reads

$$\begin{aligned}
&e_c^*(p_1^{'}, J_1^{'}, m_1^{'})e_c^*(p_2^{'}, J_2^{'}, m_2^{'})\cdots e_c(p_1, J_1, m_1)e_c(p_2, J_2, m_2)\cdots\times \\
&\times\Gamma(p_1^{'}, p_2^{'}, \cdots, p_1, p_2, \cdots, g^{\mu\nu}, \varepsilon^{\alpha\beta\gamma\delta}) \\
= \;& \eta_1^{'*}\eta_2^{'*}\cdots\eta_1\eta_2\cdots e_c^*(\bar{p}_1^{'}, J_1^{'}, m_1^{'})e_c^*(\bar{p}_2^{'}, J_2^{'}, m_2^{'})\cdots e_c(\bar{p}_1, J_1, m_1)e_c(\bar{p}_2, J_2, m_2)\cdots\times \\
&\times\Gamma(\bar{p}_1^{'}, \bar{p}_2^{'}, \cdots, \bar{p}_1, \bar{p}_2, \cdots, g^{\mu\nu}, \varepsilon^{\alpha\beta\gamma\delta}).
\end{aligned} \tag{88}$$

From Eq. (24) one can see

$$e_c^{\mu}(\bar{p}, m) = -\bar{e}_c^{\mu}(p, m), \tag{89}$$

and for spin-j canonical wave functions,

$$e_c^{\mu_1\mu_2\cdots\mu_j}(\bar{p}, j, m) = (-1)^j\bar{e}_c^{\mu_1\mu_2\cdots\mu_j}(p, j, m). \tag{90}$$

The requirement of parity conservation becomes

$$\begin{aligned}
&e_c^*(p_1^{'}, J_1^{'}, m_1^{'})e_c^*(p_2^{'}, J_2^{'}, m_2^{'})\cdots e_c(p_1, J_1, m_1)e_c(p_2, J_2, m_2)\cdots\times \\
&\times\Gamma(p_1^{'}, p_2^{'}, \cdots, p_1, p_2, \cdots, g^{\mu\nu}, \varepsilon^{\alpha\beta\gamma\delta}) \\
= \;& \eta_1^{'*}\eta_2^{'*}\cdots\eta_1\eta_2\cdots(-1)^{J_1+J_2+\cdots-J_1^{'}-J_2^{'}-\cdots}e_c^*(p_1^{'}, J_1^{'}, m_1^{'})e_c^*(p_2^{'}, J_2^{'}, m_2^{'})\cdots e_c(p_1, J_1, m_1)e_c(p_2, J_2, m_2)\cdots\times \\
&\times\Gamma(p_1^{'}, p_2^{'}, \cdots, p_1, p_2, \cdots, \bar{g}^{\mu\nu}, \bar{\varepsilon}^{\alpha\beta\gamma\delta})
\end{aligned} \tag{91}$$

or

$$\begin{aligned}
&\Gamma(p_1^{'}, p_2^{'}, \cdots, p_1, p_2, \cdots, g^{\mu\nu}, \varepsilon^{\alpha\beta\gamma\delta}) \\
= \;& \eta_1^{'*}\eta_2^{'*}\cdots\eta_1\eta_2\cdots(-1)^{J_1+J_2+\cdots-J_1^{'}-J_2^{'}-\cdots}\Gamma(p_1^{'}, p_2^{'}, \cdots, p_1, p_2, \cdots, g^{\mu\nu}, -\varepsilon^{\alpha\beta\gamma\delta}).
\end{aligned} \tag{92}$$

That is, if space reflection parity is conserved, the effective vertexes will consist of only tensors for the case

$$\eta_1^{'*}\eta_2^{'*}\cdots\eta_1\eta_2\cdots(-1)^{J_1+J_2+\cdots-J_1^{'}-J_2^{'}-\cdots}=1, \tag{93}$$

and only pseudo-tensors if

$$\eta_1^{'*}\eta_2^{'*}\cdots\eta_1\eta_2\cdots(-1)^{J_1+J_2+\cdots-J_1^{'}-J_2^{'}-\cdots}=-1. \tag{94}$$

Mixing of tensor and pseudo-tensor vertexes always means violation of space reflection symmetry. One can also prove the same result using helicity wave functions [22].

We can see from Eq. (58) that, if space reflection parity is conserved, a spin-j particle decaying into two scalar(or pseudo-scalar) particles must has the parity of $(-1)^j$, while a particle decaying into a scalar and a pseudo-scalar particle has the parity of $-(-1)^j$. Similarly, for spin-0 $\rightarrow$ spin-j + spin-0, the parity of the spin-j particle will be $(-1)^j$ (or $-(-1)^j$) if the two spin-0 particles have the same (or opposite) parities.

V. BOSON SYMMETRY

For two identical bosons, we have

$$\left[a^\dagger(\vec{p},j,\sigma),a^\dagger(\vec{p}^{'},j,\sigma^{'})\right]=0, \tag{95}$$

$$\left|\cdots;\vec{p},j,\sigma;\cdots;\vec{p}^{'},j,\sigma^{'};\cdots\right\rangle=\left|\cdots;\vec{p}^{'},j,\sigma^{'};\cdots;\vec{p},j,\sigma;\cdots\right\rangle. \tag{96}$$

This demands[3]

$$\Gamma^{\cdots\mu_1\mu_2\cdots\mu_j\cdots\nu_1\nu_2\cdots\nu_j\cdots}=\Gamma^{\cdots\nu_1\nu_2\cdots\nu_j\cdots\mu_1\mu_2\cdots\mu_j\cdots}. \tag{97}$$

Suppose particle 1 and particle 2 are identical particles. We define

$$p=p_1+p_2,\ k=p_1-p_2, \tag{98}$$

$$A_+^{\alpha\beta\gamma}=k_\mu\varepsilon^{\mu\alpha\beta\gamma}, \tag{99}$$
$$A_-^{\alpha\beta\gamma}=p_\mu\varepsilon^{\mu\alpha\beta\gamma}. \tag{100}$$

Let's first separate the 3-leg effective vertexes given in Sec. III into 1-2 symmetric parts $\Gamma_\pm^{(S)}$ and 1-2 anti-symmetric parts $\Gamma_\pm^{(A)}$:

- $(J_1,J_2,J_3)=(0,0,j)$, with j an even number

$$\Gamma_+^{(S)}=c_1(k^{\gamma_1}k^{\gamma_2}\cdots k^{\gamma_j}), \tag{101}$$
$$\Gamma_+^{(A)}=0, \tag{102}$$
$$\Gamma_-^{(S,A)}=0. \tag{103}$$

- $(J_1,J_2,J_3)=(0,0,j)$, with j an odd number

  Similar to the previous case, with $\Gamma_\pm^{(S)}\longleftrightarrow\Gamma_\pm^{(A)}$:

$$\Gamma_+^{(S)}=0, \tag{104}$$
$$\Gamma_+^{(A)}=c_1(k^{\gamma_1}k^{\gamma_2}\cdots k^{\gamma_j}), \tag{105}$$
$$\Gamma_-^{(S,A)}=0. \tag{106}$$

---

[3]Strictly speaking, we should use Green functions to derive properties of off shell effective vertexes.

- $(J_1, J_2, J_3) = (j, j, 0)$, with $j \geq 1$

$$\begin{aligned}\Gamma_+^{(S)} \;=\; & \;\; c_1 \left(g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \cdots g^{\alpha_j\beta_j}\right) \\ & + c_2 \left(g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \cdots g^{\alpha_{j-1}\beta_{j-1}} p^{\alpha_j\beta_j}\right) \\ & + \cdots \\ & + c_{j+1} \left(p^{\alpha_1} p^{\alpha_2} \cdots p^{\alpha_j} p^{\beta_1} p^{\beta_2} \cdots p^{\beta_j}\right),\end{aligned} \tag{107}$$

$$\Gamma_+^{(A)} = 0, \tag{108}$$

$$\begin{aligned}\Gamma_-^{(S)} \;=\; Q^{\alpha_1\beta_1} \; & \left\{ \; c_{j+2} \left(g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} \cdots g^{\alpha_j\beta_j}\right) \right. \\ & + c_{j+3} \left(g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} \cdots g^{\alpha_{j-1}\beta_{j-1}} p^{\alpha_j} p^{\beta_j}\right) \\ & + \cdots \\ & \left. + c_{2j+1} \left(p^{\alpha_1} p^{\alpha_2} \cdots p^{\alpha_j} p^{\beta_1} p^{\beta_2} \cdots p^{\beta_j}\right) \right\},\end{aligned} \tag{109}$$

$$\Gamma_-^{(A)} = 0. \tag{110}$$

- $(J_1, J_2, J_3) = (1, 1, j)$, with j an even number and $j \geq 2$

$$\begin{aligned}\Gamma_+^{(S)} \;=\; & \;\; c_1 g^{\alpha_1\gamma_1} g^{\beta_1\gamma_2} \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j}\right) \\ & + c_2 \left(g^{\alpha_1\gamma_1} p^{\beta_1} - g^{\beta_1\gamma_1} p^{\alpha_1}\right) \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j}\right) \\ & + c_3 g^{\alpha_1\beta_1} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j}\right) \\ & + c_4 p^{\alpha_1} p^{\beta_1} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j}\right),\end{aligned} \tag{111}$$

$$\Gamma_+^{(A)} = c_5 \left(g^{\alpha_1\gamma_1} p^{\beta_1} + g^{\beta_1\gamma_1} p^{\alpha_1}\right) \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j}\right), \tag{112}$$

$$\begin{aligned}\Gamma_-^{(S)} \;=\; & \;\; c_6 A_-^{\alpha_1\beta_1\gamma_1} \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j}\right) \\ & + c_7 Q^{\alpha_1\beta_1} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j}\right),\end{aligned} \tag{113}$$

$$\begin{aligned}\Gamma_-^{(A)} \;=\; & \;\; c_8 \left(Q^{\alpha_1\gamma_1} g^{\beta_1\gamma_2} + Q^{\beta_1\gamma_1} g^{\alpha_1\gamma_2}\right) \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j}\right) \\ & + c_9 A_+^{\alpha_1\beta_1\gamma_1} \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j}\right).\end{aligned} \tag{114}$$

- $(J_1, J_2, J_3) = (1, 1, j)$, with j an odd number and $j \geq 3$
  Similar to the previous case. Just interchange the 1-2 symmetric vertexes with the 1-2 anti-symmetric ones.
- $(J_1, J_2, J_3) = (j, j, 1)$, with $j \geq 1$

$$\begin{aligned}\Gamma_+^{(S)} \;=\; \left(g^{\alpha_1\gamma_1} p^{\beta_1} + g^{\beta_1\gamma_1} p^{\alpha_1}\right) \; & \left\{ \; c_1 \left(g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} \cdots g^{\alpha_j\beta_j}\right) \right. \\ & + c_2 \left(g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} \cdots g^{\alpha_{j-1}\beta_{j-1}} p^{\alpha_j} p^{\beta_j}\right) \\ & + \cdots \\ & \left. + c_j \left(p^{\alpha_2} p^{\alpha_3} \cdots p^{\alpha_j} p^{\beta_2} p^{\beta_3} \cdots p^{\beta_j}\right) \right\},\end{aligned} \tag{115}$$

$$\begin{aligned}\Gamma_+^{(A)} \;=\; \left(g^{\alpha_1\gamma_1} p^{\beta_1} - g^{\beta_1\gamma_1} p^{\alpha_1}\right) \; & \left\{ \; c_{j+1} \left(g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} \cdots g^{\alpha_j\beta_j}\right) \right. \\ & + c_{j+2} \left(g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} \cdots g^{\alpha_{j-1}\beta_{j-1}} p^{\alpha_j} p^{\beta_j}\right) \\ & + \cdots \\ & \left. + c_{2j} \left(p^{\alpha_2} p^{\alpha_3} \cdots p^{\alpha_j} p^{\beta_2} p^{\beta_3} \cdots p^{\beta_j}\right) \right\} \\ + k^{\gamma_1} \; & \left\{ c_{2j+1} \left(g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \cdots g^{\alpha_j\beta_j}\right) \right. \\ & + c_{2j+2} \left(g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \cdots g^{\alpha_{j-1}\beta_{j-1}} p^{\alpha_j} p^{\beta_j}\right) \\ & + \cdots \\ & \left. + c_{3j+1} \left(p^{\alpha_1} p^{\alpha_2} \cdots p^{\alpha_j} p^{\beta_1} p^{\beta_2} \cdots p^{\beta_j}\right) \right\},\end{aligned} \tag{116}$$

$$\begin{aligned}\Gamma_-^{(S)} = A_+^{\alpha_1\beta_1\gamma_1} & \left\{ c_{3j+2}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\ & +c_{3j+3}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\ & +\cdots \\ & \left.+c_{4j+1}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\},\end{aligned} \tag{117}$$

$$\begin{aligned}\Gamma_-^{(A)} = \quad & A_-^{\alpha_1\beta_1\gamma_1} \left\{ c_{4j+2}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\ & +c_{4j+3}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\ & +\cdots \\ & \left.+c_{5j+1}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\} \\ & +Q^{\alpha_1\beta_1}k^{\gamma_1} \left\{ c_{5j+2}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\ & +c_{5j+3}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\ & +\cdots \\ & \left.+c_{6j+1}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\}.\end{aligned} \tag{118}$$

- $(J_1, J_2, J_3) = (2, 2, 2)$

$$\begin{aligned}\Gamma_+^{(S)} = \; & c_1 g^{\alpha_1\beta_1}g^{\alpha_2\beta_2}k^{\gamma_1}k^{\gamma_2} \\ & +c_2 g^{\alpha_1\beta_1}g^{\alpha_2\gamma_1}g^{\beta_2\gamma_2} \\ & +c_3 g^{\alpha_1\beta_1}\left(g^{\alpha_2\gamma_1}p^{\beta_2}-g^{\beta_2\gamma_1}p^{\alpha_2}\right)k^{\gamma_2} \\ & +c_4 g^{\alpha_1\beta_1}p^{\alpha_2}p^{\beta_2}k^{\gamma_1}k^{\gamma_2} \\ & +c_5\left(g^{\alpha_1\gamma_1}g^{\alpha_2\gamma_2}p^{\beta_1}p^{\beta_2}+g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}p^{\alpha_1}p^{\alpha_2}\right) \\ & +c_6 g^{\alpha_1\gamma_1}g^{\beta_1\gamma_2}p^{\alpha_2}p^{\beta_2} \\ & +c_7 p^{\alpha_1}p^{\alpha_2}p^{\beta_1}p^{\beta_2}k^{\gamma_1}k^{\gamma_2},\end{aligned} \tag{119}$$

$$\begin{aligned}\Gamma_+^{(A)} = \; & c_8 g^{\alpha_1\beta_1}\left(g^{\alpha_2\gamma_1}p^{\beta_2}+g^{\beta_2\gamma_1}p^{\alpha_2}\right)k^{\gamma_2} \\ & +c_9\left(g^{\alpha_1\gamma_1}g^{\alpha_2\gamma_2}p^{\beta_1}p^{\beta_2}-g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}p^{\alpha_1}p^{\alpha_2}\right) \\ & +c_{10}\left(g^{\alpha_1\gamma_1}p^{\beta_1}+g^{\beta_1\gamma_1}p^{\alpha_1}\right)p^{\alpha_2}p^{\beta_2}k^{\gamma_2},\end{aligned} \tag{120}$$

$$\begin{aligned}\Gamma_-^{(S)} = \; & c_{11} g^{\alpha_1\beta_1}A_-^{\alpha_2\beta_2\gamma_1}k^{\gamma_2} \\ & +c_{12} g^{\alpha_1\beta_1}Q^{\alpha_2\beta_2}k^{\gamma_1}k^{\gamma_2} \\ & +c_{13} A_-^{\alpha_1\beta_1\gamma_1}\left(p^{\alpha_2}g^{\beta_2\gamma_2}-p^{\beta_2}g^{\alpha_2\gamma_2}\right) \\ & +c_{14} A_+^{\alpha_1\beta_1\gamma_1}\left(p^{\alpha_2}g^{\beta_2\gamma_2}+p^{\beta_2}g^{\alpha_2\gamma_2}\right) \\ & +c_{15} Q^{\alpha_1\beta_1}p^{\alpha_2}p^{\beta_2}k^{\gamma_1}k^{\gamma_2},\end{aligned} \tag{121}$$

$$\begin{aligned}\Gamma_-^{(A)} = \; & c_{16} g^{\alpha_1\beta_1}\left(Q^{\alpha_2\gamma_1}g^{\beta_2\gamma_2}+Q^{\beta_2\gamma_1}g^{\alpha_2\gamma_2}\right) \\ & +c_{17} g^{\alpha_1\beta_1}A_+^{\alpha_2\beta_2\gamma_1}k^{\gamma_2} \\ & +c_{18} A_-^{\alpha_1\beta_1\gamma_1}\left(p^{\alpha_2}g^{\beta_2\gamma_2}+p^{\beta_2}g^{\alpha_2\gamma_2}\right) \\ & +c_{19} A_+^{\alpha_1\beta_1\gamma_1}p^{\alpha_2}p^{\beta_2}k^{\gamma_2}.\end{aligned} \tag{122}$$

- $(J_1, J_2, J_3) = (2, 2, 3)$

$$\begin{aligned}\Gamma_+^{(S)} = \; & c_1 g^{\alpha_1\beta_1}\left(g^{\alpha_2\gamma_1}p^{\beta_2}+g^{\beta_2\gamma_1}p^{\alpha_2}\right)k^{\gamma_2}k^{\gamma_3} \\ & +c_2 g^{\alpha_1\gamma_1}g^{\beta_1\gamma_2}\left(g^{\alpha_2\gamma_3}p^{\beta_2}+g^{\beta_2\gamma_2}p^{\alpha_2}\right) \\ & +c_3\left(g^{\alpha_1\gamma_1}g^{\alpha_2\gamma_2}p^{\beta_1}p^{\beta_2}-g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}p^{\alpha_1}p^{\alpha_2}\right)k^{\gamma_3} \\ & +c_4\left(g^{\alpha_1\gamma_1}p^{\beta_1}+g^{\beta_1\gamma_1}p^{\alpha_1}\right)p^{\alpha_2}p^{\beta_2}k^{\gamma_2}k^{\gamma_3},\end{aligned} \tag{123}$$

$$\begin{aligned}\Gamma_+^{(A)} = \; & c_5 g^{\alpha_1\beta_1}g^{\alpha_2\beta_2}k^{\gamma_1}k^{\gamma_2}k^{\gamma_3} \\ & +c_6 g^{\alpha_1\beta_1}g^{\alpha_2\gamma_1}g^{\beta_2\gamma_2}k^{\gamma_3} \\ & +c_7 g^{\alpha_1\beta_1}\left(g^{\alpha_2\gamma_1}p^{\beta_2}-g^{\beta_2\gamma_1}p^{\alpha_2}\right)k^{\gamma_2}k^{\gamma_3} \\ & +c_8 g^{\alpha_1\beta_1}p^{\alpha_2}p^{\beta_2}k^{\gamma_1}k^{\gamma_2}k^{\gamma_3} \\ & +c_9 g^{\alpha_1\gamma_1}g^{\beta_1\gamma_2}\left(g^{\alpha_2\gamma_3}p^{\beta_2}-g^{\beta_2\gamma_2}p^{\alpha_2}\right) \\ & +c_{10}\left(g^{\alpha_1\gamma_1}g^{\alpha_2\gamma_2}p^{\beta_1}p^{\beta_2}+g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}p^{\alpha_1}p^{\alpha_2}\right)k^{\gamma_3} \\ & +c_{11} g^{\alpha_1\gamma_1}g^{\beta_1\gamma_2}p^{\alpha_2}p^{\beta_2}k^{\gamma_3} \\ & +c_{12} p^{\alpha_1}p^{\alpha_2}p^{\beta_1}p^{\beta_2}k^{\gamma_1}k^{\gamma_2}k^{\gamma_3},\end{aligned} \tag{124}$$

$$
\begin{aligned}
\Gamma_-^{(S)} = & \quad c_{13} g^{\alpha_1\beta_1} A_+^{\alpha_1\beta_2\gamma_1} k^{\gamma_2} k^{\gamma_3} \\
& + c_{14} g^{\alpha_1\beta_1} \left(Q^{\alpha_2\gamma_1} g^{\beta_2\gamma_2} + Q^{\beta_2\gamma_1} g^{\alpha_2\gamma_2} \right) k^{\gamma_3} \\
& + c_{15} A_+^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} g^{\beta_2\gamma_3} \\
& + c_{16} A_-^{\alpha_1\beta_1\gamma_1} \left(g^{\alpha_2\gamma_2} p^{\beta_2} + g^{\beta_2\gamma_2} p^{\alpha_2} \right) k^{\gamma_3} \\
& + c_{17} A_+^{\alpha_1\beta_1\gamma_1} p^{\alpha_2} p^{\beta_2} k^{\gamma_2} k^{\gamma_3},
\end{aligned} \tag{125}
$$

$$
\begin{aligned}
\Gamma_-^{(A)} = & \quad c_{18} g^{\alpha_1\beta_1} A_-^{\alpha_2\beta_2\gamma_1} k^{\gamma_2} k^{\gamma_3} \\
& + c_{19} g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} k^{\gamma_1} k^{\gamma_2} k^{\gamma_3} \\
& + c_{20} A_-^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} g^{\beta_2\gamma_3} \\
& + c_{21} A_-^{\alpha_1\beta_1\gamma_1} \left(g^{\alpha_2\gamma_2} p^{\beta_2} - g^{\beta_2\gamma_2} p^{\alpha_2} \right) k^{\gamma_3} \\
& + c_{22} A_+^{\alpha_1\beta_1\gamma_1} \left(g^{\alpha_2\gamma_2} p^{\beta_2} + g^{\beta_2\gamma_2} p^{\alpha_2} \right) k^{\gamma_3} \\
& + c_{23} Q^{\alpha_1\beta_1} p^{\alpha_2} p^{\beta_2} k^{\gamma_1} k^{\gamma_2} k^{\gamma_3}.
\end{aligned} \tag{126}
$$

- $(J_1, J_2, J_3) = (2, 2, j)$, with j an even number and $j \geq 4$

$$
\begin{aligned}
\Gamma_+^{(S)} = & \quad c_1 g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j} \right) \\
& + c_2 g^{\alpha_1\beta_1} g^{\alpha_2\gamma_1} g^{\beta_2\gamma_2} \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_3 g^{\alpha_1\beta_1} \left(g^{\alpha_2\gamma_1} p^{\beta_2} - g^{\beta_2\gamma_1} p^{\alpha_2} \right) \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j} \right) \\
& + c_4 g^{\alpha_1\beta_1} p^{\alpha_2} p^{\beta_2} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j} \right) \\
& + c_5 g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} g^{\beta_1\gamma_3} g^{\beta_2\gamma_4} \left(k^{\gamma_5} k^{\gamma_6} \cdots k^{\gamma_j} \right) \\
& + c_6 g^{\alpha_1\gamma_1} g^{\beta_1\gamma_2} \left(g^{\alpha_2\gamma_3} p^{\beta_2} - g^{\beta_2\gamma_3} p^{\alpha_2} \right) \left(k^{\gamma_4} k^{\gamma_5} \cdots k^{\gamma_j} \right) \\
& + c_7 g^{\alpha_1\gamma_1} g^{\beta_1\gamma_2} p^{\alpha_2} p^{\beta_2} \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_8 \left(g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} p^{\beta_1} p^{\beta_2} + g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} p^{\alpha_1} p^{\alpha_2} \right) \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_9 p^{\alpha_1} p^{\alpha_2} p^{\beta_1} p^{\beta_2} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j} \right),
\end{aligned} \tag{127}
$$

$$
\begin{aligned}
\Gamma_+^{(A)} = & \quad c_{10} g^{\alpha_1\beta_1} \left(g^{\alpha_2\gamma_1} p^{\beta_2} + g^{\beta_2\gamma_1} p^{\alpha_2} \right) \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j} \right) \\
& + c_{11} g^{\alpha_1\gamma_1} g^{\beta_1\gamma_2} \left(g^{\alpha_2\gamma_3} p^{\beta_2} + g^{\beta_2\gamma_3} p^{\alpha_2} \right) \left(k^{\gamma_4} k^{\gamma_5} \cdots k^{\gamma_j} \right) \\
& + c_{12} \left(g^{\alpha_1\gamma_1} g^{\alpha_2\gamma_2} p^{\beta_1} p^{\beta_2} - g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} p^{\alpha_1} p^{\alpha_2} \right) \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_{13} \left(g^{\alpha_1\gamma_1} p^{\beta_1} + g^{\beta_1\gamma_1} p^{\alpha_1} \right) p^{\alpha_2} p^{\beta_2} \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j} \right),
\end{aligned} \tag{128}
$$

$$
\begin{aligned}
\Gamma_-^{(S)} = & \quad c_{14} g^{\alpha_1\beta_1} A_-^{\alpha_2\beta_2\gamma_1} \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j} \right) \\
& + c_{15} g^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j} \right) \\
& + c_{16} A_-^{\alpha_1\beta_1\gamma_1} g^{\alpha_2\gamma_2} g^{\beta_2\gamma_3} \left(k^{\gamma_4} k^{\gamma_5} \cdots k^{\gamma_j} \right) \\
& + c_{17} A_+^{\alpha_1\beta_1\gamma_1} \left(g^{\alpha_2\gamma_2} p^{\beta_2} + g^{\beta_2\gamma_2} p^{\alpha_2} \right) \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_{18} A_-^{\alpha_1\beta_1\gamma_1} \left(g^{\alpha_2\gamma_2} p^{\beta_2} - g^{\beta_2\gamma_2} p^{\alpha_2} \right) \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_{19} Q^{\alpha_1\beta_1} p^{\alpha_2} p^{\beta_2} \left(k^{\gamma_1} k^{\gamma_2} \cdots k^{\gamma_j} \right),
\end{aligned} \tag{129}
$$

$$
\begin{aligned}
\Gamma_-^{(A)} = & \quad c_{20} g^{\alpha_1\beta_1} A_+^{\alpha_2\beta_2\gamma_1} \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j} \right) \\
& + c_{21} g^{\alpha_1\beta_1} \left(Q^{\alpha_2\gamma_1} g^{\beta_2\gamma_2} + Q^{\beta_2\gamma_1} g^{\alpha_2\gamma_2} \right) \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_{22} A_+^{\alpha_1\beta_1\gamma_1} g^{\alpha_1\gamma_2} g^{\beta_2\gamma_3} \left(k^{\gamma_4} k^{\gamma_5} \cdots k^{\gamma_j} \right) \\
& + c_{23} A_-^{\alpha_1\beta_1\gamma_1} \left(g^{\alpha_2\gamma_2} p^{\beta_2} + g^{\beta_2\gamma_2} p^{\alpha_2} \right) \left(k^{\gamma_3} k^{\gamma_4} \cdots k^{\gamma_j} \right) \\
& + c_{24} A_+^{\alpha_1\beta_1\gamma_1} p^{\alpha_2} p^{\beta_2} \left(k^{\gamma_2} k^{\gamma_3} \cdots k^{\gamma_j} \right) \\
& + c_{25} \left(Q^{\alpha_1\gamma_1} g^{\beta_1\gamma_2} + Q^{\beta_1\gamma_1} g^{\alpha_1\gamma_2} \right) g^{\alpha_2\gamma_3} g^{\beta_2\gamma_4} \left(k^{\gamma_5} k^{\gamma_6} \cdots k^{\gamma_j} \right).
\end{aligned} \tag{130}
$$

- $(J_1, J_2, J_3) = (2, 2, j)$, with j an odd number and $j \geq 5$

  Similar to the previous case, with $\Gamma_\pm^{(S)} \longleftrightarrow \Gamma_\pm^{(A)}$.

- $(J_1, J_2, J_3) = (j, j, 2)$, with $j \geq 3$

$$
\begin{aligned}
\Gamma_+^{(S)} \; = \; & \left(g^{\alpha_1\gamma_1}g^{\alpha_2\gamma_2}p^{\beta_1}p^{\beta_2}+g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}p^{\alpha_1}p^{\alpha_2}\right)\times \\
& \left\{c_1\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_2\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{j-1}\left(p^{\alpha_3}p^{\alpha_4}\cdots p^{\alpha_j}p^{\beta_3}p^{\beta_4}\cdots p^{\beta_j}\right)\right\} \\
& +g^{\alpha_1\gamma_1}g^{\beta_1\gamma_2}\times \\
& \left\{c_j\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{j+1}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{2j-1}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\} \\
& \left(g^{\alpha_1\gamma_1}p^{\beta_1}+g^{\beta_1\gamma_1}p^{\alpha_1}\right)k^{\gamma_2}\times \\
& \left\{c_{2j}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{2j+1}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& \cdots \\
& \left.+c_{3j-1}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\} \\
& +c_{3j}k^{\gamma_1}k^{\gamma_2}\left(g^{\alpha_1\beta_1}g^{\alpha_2\beta_2}\cdots g^{\alpha_j\beta_j}\right) \\
& +c_{3j+1}k^{\gamma_1}k^{\gamma_2}\left(p^{\alpha_1}p^{\alpha_2}\cdots p^{\alpha_j}p^{\beta_1}p^{\beta_2}\cdots p^{\beta_j}\right),
\end{aligned} \tag{131}
$$

$$
\begin{aligned}
\Gamma_+^{(A)} \; = \; & \left(g^{\alpha_1\gamma_1}g^{\alpha_2\gamma_2}p^{\beta_1}p^{\beta_2}-g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}p^{\alpha_1}p^{\alpha_2}\right)\times \\
& \left\{c_{3j+2}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{3j+3}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{4j}\left(p^{\alpha_3}p^{\alpha_4}\cdots p^{\alpha_j}p^{\beta_3}p^{\beta_4}\cdots p^{\beta_j}\right)\right\} \\
& \left(g^{\alpha_1\gamma_1}p^{\beta_1}-g^{\beta_1\gamma_1}p^{\alpha_1}\right)k^{\gamma_2}\times \\
& \left\{c_{4j+1}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{4j+2}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& \cdots \\
& \left.+c_{5j}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\},
\end{aligned} \tag{132}
$$

$$
\begin{aligned}
\Gamma_-^{(S)} \; = \; & A_-^{\alpha_1\beta_1\gamma_1}\left(p^{\alpha_2}g^{\beta_2\gamma_2}-p^{\beta_2}g^{\alpha_2\gamma_2}\right)\times \\
& \left\{c_{5j+1}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{5j+2}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{6j-1}\left(p^{\alpha_3}p^{\alpha_4}\cdots p^{\alpha_j}p^{\beta_3}p^{\beta_4}\cdots p^{\beta_j}\right)\right\} \\
& +A_+^{\alpha_1\beta_1\gamma_1}\left(p^{\alpha_2}g^{\beta_2\gamma_2}+p^{\beta_2}g^{\alpha_2\gamma_2}\right)\times \\
& \left\{c_{6j}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{6j+1}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{7j-2}\left(p^{\alpha_3}p^{\alpha_4}\cdots p^{\alpha_j}p^{\beta_3}p^{\beta_4}\cdots p^{\beta_j}\right)\right\} \\
& +c_{7j-1}A_-^{\alpha_1\beta_1\gamma_1}k^{\gamma_2}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right) \\
& +Q^{\alpha_1\beta_1}k^{\gamma_1}k^{\gamma_2}\times \\
& \left\{c_{7j}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{7j+1}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{8j-1}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\},
\end{aligned} \tag{133}
$$

$$
\begin{aligned}
\Gamma_{-}^{(A)} \;=\;& A_{-}^{\alpha_1\beta_1\gamma_1}\left(p^{\alpha_2}g^{\beta_2\gamma_2}+p^{\beta_2}g^{\alpha_2\gamma_2}\right)\times \\
& \left\{c_{8j}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{8j+1}\left(g^{\alpha_3\beta_3}g^{\alpha_4\beta_4}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{9j-2}\left(p^{\alpha_3}p^{\alpha_4}\cdots p^{\alpha_j}p^{\beta_3}p^{\beta_4}\cdots p^{\beta_j}\right)\right\} \\
& +A_{+}^{\alpha_1\beta_1\gamma_1}k^{\gamma_2}\times \\
& \left\{c_{9j-1}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)\right. \\
& +c_{9j}\left(g^{\alpha_2\beta_2}g^{\alpha_3\beta_3}\cdots g^{\alpha_{j-1}\beta_{j-1}}p^{\alpha_j}p^{\beta_j}\right) \\
& +\cdots \\
& \left.+c_{10j-2}\left(p^{\alpha_2}p^{\alpha_3}\cdots p^{\alpha_j}p^{\beta_2}p^{\beta_3}\cdots p^{\beta_j}\right)\right\} \\
& +c_{10j-1}\left(Q^{\alpha_1\gamma_1}g^{\beta_1\gamma_2}+Q^{\beta_1\gamma_1}g^{\alpha_1\gamma_2}\right)\left(g^{\alpha_2\beta 2}g^{\alpha_3\beta_3}\cdots g^{\alpha_j\beta_j}\right)
\end{aligned}
\tag{134}
$$

and so on.

We need to consider three cases:

1. Both of the two identical particles are internal lines

   According to Eq. (97), one can obtain

$$
\Gamma_{\pm}=\Gamma_{\pm}^{(S)}+(p_1^2-p_2^2)\Gamma_{\pm}^{(A)} \tag{135}
$$

   with the coefficients satisfying

$$
c_i=c_i\left(p_1^2+p_2^2,(p_1^2-p_2^2)^2,p_1\cdot p_2\right). \tag{136}
$$

2. One is on shell, another an internal line

   The factor $(p_1^2-W^2)$ or $(W^2-p_2^2)$ will eliminate the denominator of the propagator for the off shell particle. This make the contribution from $\Gamma_{\pm}^{(A)}$ has no pole at the particle's mass, so that it can be absorbed K. S. freely into background amplitudes.

$$
\Gamma_{\pm}=\Gamma_{\pm}^{(S)}, \tag{137}
$$

$$
c_i=c_i\left(p_1^2,p_1\cdot p_2\right). \tag{138}
$$

3. Both of the two identical particles are on shell

   Since $p_1^2-p_2^2=0$, we have

$$
\Gamma_{\pm}=\Gamma_{\pm}^{(S)} \tag{139}
$$

   with the coefficients

$$
c_i=c_i\left(p_1\cdot p_2\right). \tag{140}
$$

From Eqs. (104,106) one can infer that a particle decaying into two identical spin-0 particles must has a even spin. If parity is conserved, the parity of such a particle must be +1. For example, ρ^0, η, $\eta\prime$, ω, ϕ, a_1, f_1 etc. will not decay into two neutral pions[4].

Examples for incorporating boson symmetry in 4-leg vertexes can be found in Sec. VII and Ref. [22].

[4] $\rho^0\to 2\pi^0$ also violates (approximate) isospin symmetry. Since boson symmetry is an exact symmetry, this process is absolutely forbidden.

VI. COVARIANT HELICITY AMPLITUDES FOR TWO-BODY DECAYS

Helicity amplitudes for two body-decays can be write down directly using the wave functions and vertexes given in previous sections. Such amplitudes can be calculated in arbitrary reference frame. Especially they can be calculated in laboratory frame so that no Lorentz transformation is needed. However, one might still favor amplitudes in center of mass frame(CM frame). We will give some explicit results calculated in the rest frame of parent particles in this section.

Suppose a spin-J particle with momentum p decays into a spin-s and a spin-σ particle with momentum q and k. The helicity amplitude of such a process is

$$\begin{aligned}\mathcal{M}_{\delta\lambda\nu}(p,q,k) &\equiv \langle\vec{q},s,\lambda;\vec{k},\sigma,\nu|\mathbf{S}|\vec{p},J,\delta\rangle \\ &= \sum_{\lambda'\nu'} D^{s*}_{\lambda'\lambda}\left(W(L^{-1}(p),q)\right) D^{\sigma*}_{\nu'\nu}\left(W(L^{-1}(p),k)\right)\langle\vec{q}^{\,'},s,\lambda';\vec{k}^{\,'},\sigma',\nu|\mathbf{S}|\vec{0},J,\delta\rangle\end{aligned} \tag{141}$$

with

$$q'^{\alpha} = L^{-1\alpha}{}_{\beta}(p)q^{\beta}, \tag{142}$$

$$k'^{\alpha} = L^{-1\alpha}{}_{\beta}(p)k^{\beta}. \tag{143}$$

Here Eq. (11) has been used. Notice that we should use $L(p)$ defined in Eq. (10).

From Eqs. (9,12) one can rotate $\vec{q}^{\,'}$ to the direction of z-axis:

$$\mathcal{M}_{\delta\lambda\nu}(p,q,k) = \sum_{\lambda'\nu'} D^{s*}_{\lambda'\lambda}\left(W(L^{-1}(p),q)\right) D^{\sigma*}_{\nu'\nu}\left(W(L^{-1}(p),k)\right) D^{J*}_{\lambda'-\nu',\delta}(\varphi,\vartheta,0)F_{\lambda'\nu'}, \tag{144}$$

where (ϑ,φ) is the direction of $\vec{q}^{\,'}$. $F_{\lambda\nu}$ is the helicity amplitude in the rest frame of the parent particle,

$$F_{\lambda\nu} = \langle q,s,\lambda;k,\sigma,\nu|\mathbf{S}|p,J,\lambda-\nu\rangle_{CM}. \tag{145}$$

p,q,k have been redefined to their CM frame values in the above equation. We follow the convention of Chung [25] in this section:

$$\begin{aligned}(p^{\alpha}) &= (W;\ 0,\ 0,\ \ 0), \\ (q^{\alpha}) &= (q_0;\ 0,\ 0,\ \ r/2), \\ (k^{\alpha}) &= (k_0;\ 0,\ 0,\ -r/2).\end{aligned} \tag{146}$$

Eq. (144) is the relation between helicity amplitudes in laboratory frame and those in CM frame. It can be derived in a alternative way by writing down the explicit expressions for these amplitudes and use Lorentz transformation properties of wave functions in Eq. (17).

The masses of the daughter particles are m and μ,

$$W = q_0 + k_0, \tag{147}$$

$$q_0 = \sqrt{m^2 + \frac{r^2}{4}}, \tag{148}$$

$$k_0 = \sqrt{\mu^2 + \frac{r^2}{4}}. \tag{149}$$

The corresponding space reflection parity of the three particles are η_J, η_s and η_σ.

Parity conserving helicity amplitudes in CM frame satisfy [7,9]

$$F_{\lambda\nu} = \eta_J\eta_s\eta_\sigma(-1)^{J-s-\sigma}F_{-\lambda,-\nu}. \tag{150}$$

If the two daughter particles are identical, one has [1,7,9]

$$F_{\lambda\nu} = (-1)^J F_{\nu\lambda}. \tag{151}$$

Some explicit results for $F_{\lambda\nu}$ are listed below. We assume space reflection symmetry in all processes.

- Spin-1 $\longrightarrow$ spin-0 + spin-1, $\eta_J \eta_s \eta_\sigma = -1$
  We should choose pseudo-tensor effective vertexes.

$$F_{\lambda\nu} = -F_{-\lambda,-\nu}; \tag{152}$$

$$F_{01} = \frac{i}{2} c W r. \tag{153}$$

  Here c is a scalar.

- Spin-1 $\longrightarrow$ spin-0 + spin-1, $\eta_J \eta_s \eta_\sigma = +1$
  The effective vertex should be a tensor.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}; \tag{154}$$

$$F_{01} = -\frac{k_0}{\mu} c_2 + \frac{W}{4\mu} c_1 r^2, \tag{155}$$

$$F_{00} = -c_2. \tag{156}$$

  It can be applied to the process $a_1(1260) \longrightarrow \pi\rho$.

- Spin-1 $\longrightarrow$ spin-2 + spin-1, $\eta_J \eta_s \eta_\sigma = +1$
  The effective vertex is a tensor.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}; \tag{157}$$

$$F_{00} = \sqrt{\frac{2}{3}} \frac{k_0 q_0^2}{\mu m^2} c_5 + \frac{1}{2\sqrt{6}\mu m^2} \left(c_5 q_0 + c_3 k_0 q_0 W - c_2 k_0 W^2 + c_4 q_0 W^2 \right) r^2 + \frac{W}{8\sqrt{6}\mu m^2} \left(c_3 + c_1 W^2 \right) r^4, \tag{158}$$

$$F_{10} = \frac{k_0 q_0}{\sqrt{2}\mu m} c_5 + \frac{1}{4\sqrt{2}\mu m} \left(c_5 + c_4 W^2 \right) r^2, \tag{159}$$

$$F_{01} = -\frac{1}{\sqrt{6}} c_5 - \frac{W^2}{2\sqrt{6} m^2} c_2 r^2, \tag{160}$$

$$F_{11} = -\frac{q_0}{\sqrt{2} m} c_5 - \frac{W}{2\sqrt{2} m} c_3 r^2, \tag{161}$$

$$F_{21} = -c_5. \tag{162}$$

  These amplitudes are applicable to the case $J/\psi \longrightarrow a_2(1320)\rho$.

- Spin-1 $\longrightarrow$ spin-4 + spin-1, $\eta_J \eta_s \eta_\sigma = +1$
  Use the tensor vertexes given in Sec. III.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}; \tag{163}$$

$$F_{00} = \frac{k_0 q_0 W^2}{\sqrt{70} m^4} c_5 r^2 + \frac{W^2}{\sqrt{1120} m^4} \left(c_5 q_0 + c_3 k_0 q_0 W - c_2 k_0 W^2 + c_4 q_0 W^2 \right) r^4 + \frac{W^3}{\sqrt{17920} m^4} \left(c_3 + c_1 W^2 \right) r^6, \tag{164}$$

$$F_{01} = -\frac{W^2}{\sqrt{280} m^2} c_5 r^2 - \frac{W^4}{\sqrt{1120} m^4} c_4 r^4, \tag{165}$$

$$F_{10} = \frac{k_0 q_0 W^2}{4\sqrt{7} m^3} c_5 r^2 + \frac{W^2}{16\sqrt{7} m^3} \left(c_5 + c_4 W^2 \right) r^4, \tag{166}$$

$$F_{11} = -\frac{q_0 W^2}{4\sqrt{7} m^3} c_5 r^2 - \frac{W^3}{16\sqrt{7} m^3} c_3 r^4, \tag{167}$$

$$F_{21} = -\frac{W^2}{4\sqrt{7} m^2} c_5 r^2. \tag{168}$$

- Spin-0 $\longrightarrow$ spin-1 + spin-1, $\eta_J \eta_s \eta_\sigma = -1$

  The effective vertex is a pseudo scalar.

$$F_{\lambda\nu} = -F_{-\lambda,-\nu}; \tag{169}$$

$$F_{11} = igWr. \tag{170}$$

  These amplitudes automatically satisfy

$$F_{\lambda\nu} = F_{\nu\lambda}. \tag{171}$$

- Spin-0 $\longrightarrow$ spin-1 + spin-1, $\eta_J \eta_s \eta_\sigma = +1$

  The vertex is a tensor.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}; \tag{172}$$

$$F_{00} = -\frac{k_0 q_0}{m\mu} c_2 - \frac{1}{4m\mu}\left(c_2 + c_1 W^2\right) r^2, \tag{173}$$

$$F_{11} = c_2. \tag{174}$$

- Spin-0 $\longrightarrow$ spin-0 + spin-1, $\eta_J \eta_s \eta_\sigma = -1$

  The vertex should be a tensor.

$$F_{00} = -\frac{W}{2\mu} c. \tag{175}$$

- Spin-2 $\longrightarrow$ spin-0 + spin-0, $\eta_J \eta_s \eta_\sigma = +1$

  The vertex should be a tensor,

$$F_{00} = -\frac{1}{\sqrt{24}} c r^2. \tag{176}$$

  It can be applied to the decay $f_2(1270) \longrightarrow \pi^+\pi^-$.

- Spin-2 $\longrightarrow$ two identical spin-1 particles, $\eta_J = +1$

  The vertex should satisfy boson symmetry.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}, \tag{177}$$

$$F_{\lambda\nu} = F_{\nu\lambda}; \tag{178}$$

$$F_{00} = \sqrt{\frac{2}{3}} \frac{q_0^2}{m^2} c_4 + \frac{1}{\sqrt{6}m^2}\left(\frac{c_2}{2} q_0^2 - c_3 q_0 W\right) r^2 + \frac{1}{\sqrt{384}m^2}\left(c_2 + c_1 W^2\right) r^4, \tag{179}$$

$$F_{01} = \frac{q_0}{\sqrt{2}m} c_4 - \frac{W}{4\sqrt{2}m} c_3 r^2, \tag{180}$$

$$F_{1,-1} = c_4, \tag{181}$$

$$F_{11} = \frac{1}{\sqrt{6}} c_4 - \frac{1}{2\sqrt{6}} c_2 r^2. \tag{182}$$

- Spin-2 $\longrightarrow$ spin-2 + spin-0, $\eta_J\eta_s\eta_\sigma = +1$
  The effective vertex should be a tensor.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}; \tag{183}$$

$$F_{00} = \frac{1}{3}\left(1+\frac{2q_0^2}{m^2}\right)c_3 - \frac{q_0 W}{6m^2}c_2 r^2 + \frac{W^2}{24m^2}c_1 r^4, \tag{184}$$
$$F_{10} = \frac{q_0}{m}c_3 - \frac{W}{8m}c_2 r^2, \tag{185}$$
$$F_{20} = c_3. \tag{186}$$

- Spin-4 $\longrightarrow$ two identical spin-1 particles, $\eta_J = +1$
  The vertex should satisfy boson symmetry.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}, \tag{187}$$
$$F_{\lambda\nu} = F_{\nu\lambda}; \tag{188}$$

$$F_{00} = -\frac{q_0^2}{\sqrt{70}m^2}c_4 r^2 - \frac{1}{\sqrt{1120}m^2}\left(c_2 q_0^2 + 2c_3 q_0 W\right)r^4 - \frac{1}{\sqrt{17920}m^2}\left(c_2 + c_1 W^2\right)r^6, \tag{189}$$
$$F_{01} = -\frac{q_0}{4\sqrt{7}m}c_4 r^2 - \frac{W}{16\sqrt{7}m}c_3 r^4, \tag{190}$$
$$F_{1,-1} = -\frac{1}{4\sqrt{7}}c_4 r^2, \tag{191}$$
$$F_{11} = -\frac{1}{\sqrt{280}}c_4 r^2 + \frac{1}{\sqrt{1120}}c_2 r^4. \tag{192}$$

- Spin-4 $\longrightarrow$ spin-2 + spin-0, $\eta_J\eta_s\eta_\sigma = +1$
  The effective vertex is a tensor.

$$F_{\lambda\nu} = +F_{-\lambda,-\nu}; \tag{193}$$

$$F_{00} = \frac{1}{2\sqrt{105}}\left(1+\frac{2q_0^2}{m^2}\right)c_3 r^2 - \frac{W q_0}{4\sqrt{105}m^2}c_2 r^4 + \frac{W^2}{16\sqrt{105}m^2}c_1 r^6, \tag{194}$$
$$F_{10} = \frac{q_0}{\sqrt{56}m}c_3 r^2 - \frac{W}{\sqrt{3584}m}c_2 r^4, \tag{195}$$
$$F_{20} = \frac{1}{4\sqrt{7}}c_3 r^2. \tag{196}$$

- Spin-1 $\longrightarrow$ spin-0 + spin-0, $\eta_J\eta_s\eta_\sigma = -1$
  The effective vertex should be a vector.

$$F_{00} = -\frac{c}{2}r. \tag{197}$$

VII. RESONANCES AND BACKGROUNDS

For a process involving more than three particles, we must separate the vertexes into one-particle irreducible (1PI) parts and one-particle reducible (1PR) parts. Usually 1PI parts are called backgrounds, while 1PR parts are called resonances. An example with 4-leg vertex are shown in Fig. 2.

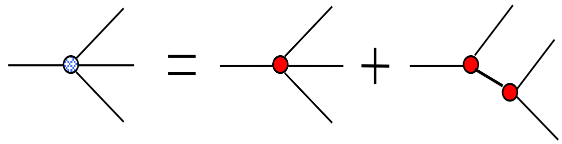

FIG. 2. A 4-leg vertex can be divided into an 1PI part and resonance parts

Feynman graphs for the process $a_1 \to \pi^+\pi^+\pi^-$ are illustrated in Fig. 3. We consider only ρ resonance here. The four-momenta of a_1 and the three pions are p, p_1, p_2 and p_3, with

$$p = p_1 + p_2 + p_3. \tag{198}$$

The corresponding spin-parities of a_1, ρ and π are 1^-, 1^- and 0^- [27].

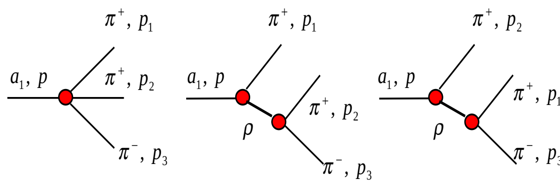

FIG. 3. The 1PI graph and two ρ-resonance graphs for the process $a_1 \to \pi^+\pi^+\pi^-$

The background amplitude can be found after an analysis similar to that in Ref. [22][5]:

$$\mathcal{M}_\lambda^{(b)} = e_\mu(p,\lambda)\{b_1(p_1+p_2)^\mu + [(p_1-p_2)\cdot p_3]b_2(p_1-p_2)^\mu\}, \tag{199}$$

$$b_i = b_i\left((p_1+p_2)\cdot p_3, [(p_1-p_2)\cdot p_3]^2\right). \tag{200}$$

We use Breit-Wigner factors as approximation to the full propagators in Fig. 3, i.e., write the propagators of ρ^0 as

$$\frac{g^{\alpha\beta} - (p_2+p_3)^\alpha (p_2+p_3)^\beta / m_\rho^2}{(p_2+p_3)^2 - m_\rho^2 + i\Gamma_\rho m_\rho} \tag{201}$$

and

$$\frac{g^{\alpha\beta} - (p_1+p_3)^\alpha (p_1+p_3)^\beta / m_\rho^2}{(p_1+p_3)^2 - m_\rho^2 + i\Gamma_\rho m_\rho}, \tag{202}$$

where m_ρ is the mass of ρ^0 and Γ_ρ its width. Alternatively, one can choose propagators in other forms to get better approximations.

The $\rho^0\pi^+\pi^-$ vertexes must be vectors if parity is conserved, which can be read out from the list of Sec. III:

$$\Gamma^\beta_{\rho\pi^+\pi^-}(p_2,p_3) = c(p_2\cdot p_3)\,(p_2-p_3)^\beta\,, \tag{203}$$

$$\Gamma^\beta_{\rho\pi^+\pi^-}(p_1,p_3) = c(p_1\cdot p_3)\,(p_1-p_3)^\beta\,. \tag{204}$$

The $a_1\pi^+\rho^0$ vertexes must be tensors,

---

[5] The c_3 term in Eq.(10) of Ref. [22] can be dropped without introducing K. S..

$$\Gamma^{\mu\alpha}_{a_1\pi^+\rho}(p_1, p_2+p_3) = c_1^{'}(p_2\cdot p_3)g^{\mu\alpha} + c_2^{'}(p_2\cdot p_3)p_1^\mu p_1^\alpha, \quad (205)$$

$$\Gamma^{\mu\alpha}_{a_1\pi^+\rho}(p_2, p_1+p_3) = c_1^{'}(p_1\cdot p_3)g^{\mu\alpha} + c_2^{'}(p_1\cdot p_3)p_2^\mu p_2^\alpha. \quad (206)$$

Combining all of these together, we find the resonance part of the amplitude to be

$$\begin{aligned} \mathcal{M}_\lambda^{(res)} = e_\mu(p,\lambda)\{ \quad & D_{23}[\tfrac{3}{2}c_1(s_{23}) + \tfrac{1}{2}p_1\cdot(p_2-p_3)c_2(s_{23})](p_1+p_2)^\mu \\ & +D_{23}[-\tfrac{1}{2}c_1(s_{23}) + \tfrac{1}{2}p_1\cdot(p_2-p_3)c_2(s_{23})](p_1-p_2)^\mu \\ & +D_{13}[\tfrac{3}{2}c_1(s_{13}) + \tfrac{1}{2}p_2\cdot(p_1-p_3)c_2(s_{13})](p_1+p_2)^\mu \\ & +D_{13}[-\tfrac{1}{2}c_1(s_{13}) + \tfrac{1}{2}p_2\cdot(p_1-p_3)c_2(s_{13})](p_2-p_1)^\mu\}. \end{aligned} \quad (207)$$

Here $p_3^\mu \simeq -(p_1+p_2)^\mu$ have been used, and

$$s_{23} = (p_2+p_3)^2, \quad (208)$$

$$s_{13} = (p_1+p_3)^2, \quad (209)$$

$$D_{23} = \frac{1}{s_{23} - m_\rho^2 + i\Gamma_\rho m_\rho}, \quad (210)$$

$$D_{13} = \frac{1}{s_{13} - m_\rho^2 + i\Gamma_\rho m_\rho}. \quad (211)$$

It is easy to see

$$p_1\cdot(p_2-p_3) = \frac{m_{a_1}^2 + 9m_\pi^2 - 2(2s_{13}+s_{23})}{2}, \quad (212)$$

$$p_2\cdot(p_1-p_3) = \frac{m_{a_1}^2 + 9m_\pi^2 - 2(s_{13}+2s_{23})}{2}. \quad (213)$$

After redefinition of c_1, c_2 and b_2, the covariant helicity amplitude becomes

$$\begin{aligned} \mathcal{M}_\lambda \equiv\ & \mathcal{M}_\lambda^{(b)} + \mathcal{M}_\lambda^{(res)} \\ =\ & e_\mu(p,\lambda)(p_1+p_2)^\mu \times \\ & \{\ b_1\left(s_{13}+s_{23}, (s_{13}-s_{23})^2\right) \\ & +3c_1(s_{23})D_{23} + 3c_1(s_{13})D_{13} \\ & +[m_{a_1}^2 + 9m_\pi^2 - 2(2s_{13}+s_{23})]c_2(s_{23})D_{23} \\ & +[m_{a_1}^2 + 9m_\pi^2 - 2(s_{13}+2s_{23})]c_2(s_{13})D_{13}\} \\ & +e_\mu(p,\lambda)(p_1-p_2)^\mu \times \\ & \{\ [s_{13}-s_{23}]b_2\left(s_{13}+s_{23}, (s_{13}-s_{23})^2\right) \\ & -c_1(s_{23})D_{23} + c_1(s_{13})D_{13} \\ & +[m_{a_1}^2 + 9m_\pi^2 - 2(2s_{13}+s_{23})]c_2(s_{23})D_{23} \\ & -[m_{a_1}^2 + 9m_\pi^2 - 2(s_{13}+2s_{23})]c_2(s_{13})D_{13}\}. \end{aligned} \quad (214)$$

The background amplitude will not give a flat distribution in the Dalitz plot of the three pions:

$$\begin{aligned} \sum_\lambda |\mathcal{M}_\lambda^{(b)}|^2 =\ & -|b_1|^2(p_1+p_2)^2 - |b_2|^2(p_1-p_2)^2[(p_1-p_2)\cdot p_3]^2 \\ & +\tfrac{1}{m_{a_1}^2}\{\ |b_1|^2[p\cdot(p_1+p_2)]^2 + |b_2|^2[p\cdot(p_1-p_2)]^2[(p_1-p_2)\cdot p_3]^2 \\ & \quad +(b_1^*b_2 + b_1b_2^*)[p\cdot(p_1+p_2)][p\cdot(p_1-p_2)][(p_1-p_2)\cdot p_3]\}. \end{aligned} \quad (215)$$

In fact for any process involving particles with non-zero spins, the background distributions are not flat. If we do not include such background(1PI) terms, those resonances terms we have considered might just simulating the background distributions. Let's take the process $a_1 \to \pi^+\pi^+\pi^-$ as an example. We can not say that we have seen $\rho\prime$ or some other resonances, if the background term $\mathcal{M}_\lambda^{(b)}$ is not considered when we fit data. Particularly, this *is* the case for resonances far off shell or with large widths. **One must include background terms**. A resonance in a process can be taken as well established under two conditions: (1) We must include the background terms in amplitudes with and without such a resonance. (2) The amplitudes including the resonance significantly improve the best fit to experimental data, comparing to those ignoring it.

VIII. SUMMARY

The main results of this paper are summarized bellow:

A list of general 3-leg effective vertexes for bosons is given, with kinematic singularities carefully avoided. Space reflection symmetry demand effect vertexes to be tensors or pseudo-tensors depending on spin-parities of external lines. Mixing of tensor and pseudo-tensor vertexes always means violation of parity conservation. Boson symmetry require that effective vertexes take the special form given in Sec.V. The requirement of parity conservation and boson symmetry leads to selection rules. These results are needed when we construct phenomenological models or write amplitudes to fit data of high energy experiments.

Helicity amplitudes in laboratory frame are related to those in center of mass frame by Wigner rotations. For two-body decays, it is possible to write the explicit expressions of covariant helicity amplitudes in a concise form.

S-matrixes for processes involving more than three particles can be divide into 1PI parts and one-particle reducible parts, or in another words, backgrounds and resonances. We emphasize that such background terms are important when one try to extract meaningful information on resonances. This is especially the case if the width of the resonances are large, or the resonances are far off shell.

Constraints of gauge invariance on effective vertexes will be the content of another paper.

ACKNOWLEDGEMENT

Jie-Jie Zhu wishes to thank Prof. Yu-Can Zhu and Dr. Liao-Yuan Dong for discussions. This work is partially supported by National Founds of China through C. N. Yang, Funds of IHEP of China and the Grant LWTZ-1298 of Chinese Academy of Science.

---

[1] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).
[2] Chou Kuang-Chao and M. I. Shirokov, J. Exp. Theor. Phys. **34**, 1230 (1958).
[3] S. M. Berman and M. Jacob, Phys. Rev. **139**, B1023 (1965).
[4] G. C. Wick, Ann. Phys. (N.Y.) **18**, 65 (1962).
[5] C. Zemach, Phys. Rev. **133**, B1201 (1964).
[6] C. Zemach, Phys. Rev. **140**, B97 (1965).
[7] S. U. Chung, Phys. Rev. D **48**, 1225 (1993).
[8] V. Filippini, A. Fontana and A. Rotondi, Phys. Rev. D **51**, 2247 (1995).
[9] S. U. Chung, *Spin Formalisms*, CERN 71-8 (1971).
[10] H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento, **10**, 205 (1955).
[11] N. N. Bogoliubov, A. A. Logunov and I. T. Todorov, *Introduction to Axiomatic Field Theory*, (Benjamin/Cummings Reading MA, 1975).
[12] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
[13] H. P. Stapp, Phys. Rev. **125**, 2139 (1962).
[14] Y. Hara, Phys. Rev. **136**, B507 (1964).
[15] H. P. Stapp, Phys. Rev. **160**, 1251 (1967).
[16] L. L. C. Wang, Phys. Rev. **142**, 1187 (1966).
[17] G. Cohen-Tannoudji, A. Morel and H. Navelet, Ann. Phys., **46**, 239 (1968).
[18] W. A. Bardeen and Wu-Ki Tung, Phys. Rev. **173**, 1423 (1968).
[19] R. W. Brown and I. J. Muzinich, Phys. Rev. D **4**, 1496 (1971).
[20] M. D. Scadron, Phys. Rev. **165**, 1640 (1968).
[21] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[22] Jie-Jie Zhu and Tu-Nan Ruan, Phys. Rev. D **57**, 5468 (1998).
[23] Jie-Jie Zhu, *thesis*, (University of Science and Technology of China, 1997), unpublished.
[24] Steven Weinberg, *The Quantum Theory of Fields, Vol. I, Foundations*, (Cambridge University Press, Cambridge, 1995).
[25] S. U. Chung, BNL Report No. BNL-QGS94-22 (1995).
[26] C. Fronsdal, Nuovo Cimento, **Suppl. 9**, 416 (1958).
[27] Particle Data Group, C. Caso *et al.*, Eur. Phys. J. C **3**, 1 (1998).